

# Heterogeneity in Decentralized Asset Markets <sup>\*</sup>

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## Abstract

We study a search and bargaining model of asset markets in which investors' heterogeneous valuations for the asset are drawn from an arbitrary distribution. We present a solution technique that makes the model fully tractable, and allows us to provide a complete characterization of the unique equilibrium, in closed-form, both in and out of steady-state. Using this characterization, we derive several novel implications that highlight the importance of heterogeneity. In particular, we show how some investors endogenously emerge as intermediaries, even though they have no advantage in contacting other agents or holding inventory; and we show how heterogeneity magnifies the impact of search frictions on asset prices, misallocation, and welfare.

**Keywords:** search frictions, bargaining, heterogeneity, price dispersion

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# 1 Introduction

In this paper, we extend the workhorse model of an unintermediated or “pure” decentralized asset market to allow for arbitrary heterogeneity over agents’ valuations. Despite its greater complexity, the model remains fully tractable: we provide a solution technique that delivers a complete characterization of the equilibrium, in closed form, both in and out of steady state. Equipped with this characterization, we derive a variety of novel implications regarding the patterns of trade in pure decentralized asset markets. We focus on two such implications that highlight the importance of heterogeneity: the concentration of trading volume across market participants with different valuations, and the impact of search frictions on the level and dispersion of asset prices.

Our starting point is the model of decentralized or “over-the-counter” (OTC) asset markets developed by [Duffie et al. \(2005, 2007\)](#), hereafter DGP, in which a fixed measure of agents, who can hold either zero or one share of an asset in fixed supply, have time-varying utility types that generate heterogeneous valuations for the asset. Agents are periodically and randomly matched in pairs, and bargain over the price if there are gains from trade. These simple ingredients form the basis of a workhorse model because they capture two important features of OTC markets that are at odds with the standard, Walrasian paradigm—namely, that it takes time to find a willing counterparty to trade, and that the terms of trade are typically determined in a bilateral fashion. As a result, the model has provided a useful tool for understanding the basic effects of search and bargaining frictions on asset prices, misallocation, liquidity, and welfare.

However, the recent arrival of high quality, transaction-level data has provided a more detailed, granular description of trading patterns that are common across many OTC markets, including the markets for corporate bonds, municipal bonds, credit derivatives, and Federal Funds loans. In particular, this data highlights the importance of heterogeneity in OTC markets, as market participants differ systematically with respect to how often they trade, the direction in which they tend to trade (buying or selling), and the prices they pay or receive. Unfortunately, following the original formulation of DGP, most of the existing literature has focused on the special case in which agents’ utility types are drawn from a two-point distribution. While this assumption buys tractability by keeping the state space finite, it also places severe limits on the model’s ability to confront these new facts from the micro data. For example, in the special case of two types, those

agents with a low valuation sell to those agents with a high valuation at the same rate and the same price—there is dispersion in neither the (expected) concentration of trading volume across agents nor in the terms of trade.

To overcome these limitations, we analyze the benchmark model of decentralized asset markets with *arbitrary* heterogeneity in agents’ valuations. Our main contribution is methodological: we obtain a closed-form characterization of the equilibrium, both in and out of the steady state. Hence, we show that extending the benchmark model to an environment with arbitrary heterogeneity in valuations implies essentially *no* loss in tractability, while delivering new theoretical insights into the effects of heterogeneity in decentralized markets, along with a richer framework to confront data emerging from various OTC markets.

In particular, after laying out the environment in Section 2, we develop the methodology that allows for our equilibrium characterization in Section 3. This requires deriving explicit solutions for the joint distributions of asset holdings and utility types, and for the market participants’ reservation values; both of these derivations are new to the literature. Moreover, in contrast to the usual guess-and-verify approach, we establish several elementary properties of reservation values directly—without making *a priori* assumptions on the direction of gains from trade—which allows us later to confirm the uniqueness of our equilibrium. Finally, as a by-product of our solution technique, we also present a sequential representation of reservation values that generalizes the concept of a marginal investor to a decentralized market: reservation values can be computed as the present value of utility flows to a hypothetical investor, with an endogenous process for their (stochastic) valuations that reflect the relevant search and bargaining frictions.

Then, in Section 4, we explore the implications of our model for OTC market outcomes. While our equilibrium characterization offers a potentially wide range of predictions, we focus on statistics that highlight the importance of heterogeneity in decentralized asset markets.

First, we study the model’s implications for the concentration of trading volume across market participants. We find that even though meetings are random in our model, the patterns of trade are not: the model implies that a small number of agents account for most of the trading volume, buying and selling relatively frequently, while the remaining agents account for a small fraction of volume, and tend to trade in only one direction. This pattern of trade, often referred to as a “core-

periphery” trading network, is observed in most OTC markets. Of course, there are many other important factors that contribute to this trading pattern in the real world, including the prevalence of long-term relationships between market participants and heterogeneity in size and trading speed, among others. However, the fact that these patterns arise without these important factors reveals an underlying gravitational pull toward a market structure in which certain agents emerge as natural intermediaries in pure decentralized markets.

Second, we study the relationship between heterogeneity, search and bargaining frictions, asset prices, and welfare. We show that heterogeneity magnifies the impact of frictions on various equilibrium outcome, and that this impact is more pronounced on price levels than on price dispersion and welfare. Hence, using observed price dispersion to quantify the effect of search frictions on price discounts or premia can be misleading: price dispersion can essentially vanish while price levels are still far from their frictionless counterpart. The welfare cost of search frictions, however, are more accurately captured by price dispersion.

## 1.1 Related Literature

This paper belongs to the literature that applies search-and-matching theory to the study of asset markets.<sup>1</sup> As is well known, the main challenge in analyzing such theoretical market settings is that the relevant state variable is an infinite-dimensional object: the joint distribution of agents’ types and portfolios. Following [Duffie, Gârleanu, and Pedersen \(2007\)](#), most of the early literature responded by restricting attention to the case of two valuations.<sup>2</sup> Our main contribution is to show that this restriction is unnecessary: one can provide a closed-form characterization of the unique equilibrium, both in and out of steady-state, with arbitrary heterogeneity in valuations.

The present paper merges and replaces two working papers, [Hugonnier \(2012\)](#) and [Lester and Weill \(2013\)](#), in which we independently developed the methods to characterize equilibria with arbitrary heterogeneity in valuations. Among the few papers that pre-date these two working papers and study pure decentralized asset markets with more than two types of investors, perhaps the most

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<sup>1</sup>Beyond asset markets, our analysis applies equally well to capital (traded in secondary markets), ideas (traded in venture capital markets), and other durable objects that trade in decentralized markets.

<sup>2</sup>The model with two valuations has been explored extensively in both the theoretical and empirical strands of the literature; see, for example, [Vayanos and Wang \(2007\)](#), [Vayanos and Weill \(2008\)](#), [Weill \(2008\)](#), [Trejos and Wright \(2014\)](#), [Chiu and Koepl \(2016\)](#), and [Gavazza \(2011a, 2016\)](#), to name just a few.

closely related to our work are [Gavazza \(2011b\)](#) and [Afonso and Lagos \(2015\)](#).<sup>3</sup> In particular, in an online appendix, [Gavazza \(2011b\)](#) proposes a model of purely decentralized trade with a continuum of types, but focuses on the case in which investors trade only once between preference shocks. In contrast, many of the insights that arise in our environment derive from the many trading opportunities that arise between preference shocks. In [Afonso and Lagos \(2015\)](#), the heterogeneity in valuations derives from allowing investors to take on arbitrary (discrete) asset positions.<sup>4</sup> While several of the insights from [Afonso and Lagos \(2015\)](#) also arise in our environment, the two papers differ in terms of both methodology and focus: while they establish many results via numerical methods in attempt to confront trading patterns in the federal funds market, we derive a variety of analytical results that allow us to study implications for volume and prices across a broad range of OTC markets.

A number of subsequent papers have explored applications of our results, as well as alternative dimensions of heterogeneity that are relevant in OTC markets, including [Shen, Wei, and Yan \(2015\)](#), [Üslü \(2019\)](#), [Sagi \(2015\)](#), [Farboodi, Jarosch, and Shimer \(2018a\)](#), [Farboodi, Jarosch, Menzio, and Wiriadinata \(2018b\)](#), [Bethune, Sultanum, and Trachter \(2018\)](#), [Zhang \(2017\)](#), [Liu \(2018\)](#), [Tse and Xu \(2018\)](#), and [Yang and Zeng \(2019\)](#). For a more comprehensive review of this, we refer the reader to [Hugonnier, Lester, and Weill \(2019\)](#). In that paper, we build on the methods developed here to study an OTC market with customers and dealers, where the dealers themselves trade in a pure decentralized market. We characterize the equilibrium, derive a number of qualitative implications, and then explore the model’s quantitative predictions using data from the municipal bond market. In addition to developing the methods that are applied in [Hugonnier, Lester, and Weill \(2019\)](#), the present paper offers a number of new results and insights. In particular, by studying a simpler model—a single search market with ex-ante identical investors—we are able to characterize the complete equilibrium in closed form, derive out-of-steady state dynamics, study the properties of equilibria as frictions vanish, and show that heterogeneity magnifies the impact of search frictions on allocations, prices, and welfare.

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<sup>3</sup>See also [Cujean and Praz \(2013\)](#) and [Neklyudov \(2019\)](#).

<sup>4</sup>In contrast, we retain the assumption that investors asset holdings lie in  $\{0, 1\}$ , and “hardwire” the heterogeneity directly into preferences.

## 2 The model

### 2.1 Preference, endowments, and matching technology

We consider a continuous-time, infinite-horizon model in which time is indexed by  $t \geq 0$ . The economy is populated by a unit measure of infinitely-lived and risk-neutral investors who discount the future at the same rate  $r > 0$ . There is one indivisible, durable asset in fixed supply,  $s \in (0, 1)$ , and one perishable good that we treat as the numéraire.

Investors can hold either zero or one unit of the asset.<sup>5</sup> The utility flow an investor receives at time  $t$  from holding a unit of the asset, which we denote by  $\delta_t$ , differs across investors and, for each investor, changes over time. In particular, each investor receives i.i.d. preference shocks that arrive according to a Poisson process with intensity  $\gamma$ , whereupon the investor draws a new utility flow  $\delta'$  from some cumulative distribution function  $F(\delta')$ .<sup>6</sup> We assume that the support of this distribution is a compact interval and make it sufficiently large so that  $F(\delta)$  has no mass points at its boundaries. For simplicity, we normalize this interval to  $[0, 1]$ . Thus, at this point, we place very few restrictions on the distribution of utility types: our solution method applies equally well to discrete distributions such as the two-point distribution of DGP, continuous distributions, and mixtures of the two.

Investors trade in a purely decentralized market in which each investor initiates contact with another randomly selected investor according to a Poisson process with intensity  $\lambda/2$ . If two investors are matched and there are gains from trade, they bargain over the price of the asset. The outcome of the bargaining game is taken to be the Nash bargaining solution, in which the investor with asset holdings  $q \in \{0, 1\}$  has bargaining power  $\theta_q \in (0, 1)$ , with  $\theta_0 + \theta_1 = 1$ .

An important object of interest throughout our analysis will be the joint distribution of utility types and asset holdings. The standard approach in the literature, following DGP, is to characterize this distribution by analyzing the density or *measure* of investors across types  $(q, \delta) \in \{0, 1\} \times [0, 1]$ . Our analysis below reveals that the model becomes much more tractable when

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<sup>5</sup>For the purpose of analyzing steady states, we could equivalently assume that investors can trade any quantity of the asset but are constrained to hold a maximum quantity that is normalized to one share: given linear utility, they would find it optimal to trade and hold either zero or one share.

<sup>6</sup>The characterization and main properties of the equilibrium remain qualitatively unchanged if we assume that utility types are persistent in the sense that, conditional on experiencing a preference shock, the probability of drawing a utility type in  $[0, \delta']$  is given by some function  $F(\delta'|\delta)$  that is decreasing in  $\delta$  for any  $\delta' \in [0, 1]$ .

we study instead the *cumulative measure*; this allows for a closed-form solution for an arbitrary underlying distribution of types, both in and out of steady state. To this end, let  $\Phi_{q,t}(\delta)$  denote the measure of investors at time  $t \geq 0$  with asset holdings  $q \in \{0, 1\}$  and utility type less than  $\delta \in [0, 1]$ . Assuming that initial types are randomly drawn from the cumulative distribution  $F(\delta)$ , the following accounting identities must hold for all  $t \geq 0$ :<sup>7</sup>

$$\Phi_{0,t}(\delta) + \Phi_{1,t}(\delta) = F(\delta) \tag{1}$$

$$\Phi_{1,t}(1) = s. \tag{2}$$

Equation (1) highlights that the cross-sectional distribution of utility types in the population is constantly equal to  $F(\delta)$ , since initial utility types are drawn from  $F(\delta)$  and investors' new types are independent from their previous types. Equation (2) is a market clearing condition that equates the total measure of investors who own the asset and the total supply of assets in the economy. Given our previous assumptions, this condition is equivalent to  $\Phi_{0,t}(1) = 1 - s$  for all  $t \geq 0$ .

## 2.2 The Frictionless Benchmark: Centralized Exchange

Consider a frictionless environment in which there is a competitive, centralized market where investors can buy or sell the asset instantly at some price  $p_t$ , which must be constant in equilibrium since the cross-sectional distribution of types in the population is time-invariant.

In such an environment, the objective of an investor is to choose a finite variation asset-holding process  $q_t \in \{0, 1\}$  that is progressively measurable with respect to the filtration generated by his utility-type process, and which maximizes

$$\mathbb{E}_{0,\delta} \left[ \int_0^\infty e^{-rt} \delta_t q_t dt - \int_0^\infty e^{-rt} p dq_t \right] = pq_0 + \mathbb{E}_{0,\delta} \left[ \int_0^\infty e^{-rt} (\delta_t - rp) q_t dt \right].$$

This representation makes it clear that, at each time  $t$ , an investor's optimal holdings satisfy

$$q_t^* = \begin{cases} 0 & \text{if } \delta_t < rp \\ 0 \text{ or } 1 & \text{if } \delta_t = rp \\ 1 & \text{if } \delta_t > rp. \end{cases}$$

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<sup>7</sup>Most of our results extend to the case in which the initial distribution is not drawn from  $F(\delta)$ . As a result, our framework could be useful for analyzing the effects of an aggregate shock that shifts (or otherwise changes) the distribution of valuations across investors. See Appendix B for more details.

This immediately implies that, in equilibrium, the asset is allocated at each time to the investors who value it most. As a result, the distribution of types among investors who own one unit of the asset is time invariant and given by

$$\Phi_1^*(\delta) = \max \{0, F(\delta) - (1 - s)\} \equiv (F(\delta) - (1 - s))^+.$$

It now follows from (1) that the distribution of utility types among investors who do not own the asset is explicitly given by  $\Phi_0^*(\delta) = \min\{F(\delta), 1 - s\}$ . The “marginal” type—i.e., the utility type of the investor who has the lowest valuation among all owners of the asset—is then defined by

$$\delta^* = \inf \{ \delta \in [0, 1] : 1 - F(\delta) \leq s \},$$

and the equilibrium price of the asset  $p^*$  has to equal  $\delta^*/r$ —i.e., the present value of the utility flows enjoyed by a hypothetical investor who holds the asset forever and whose utility type is constantly equal to the marginal type.<sup>8</sup>

### 3 Equilibrium with search frictions

We now characterize the equilibrium with search frictions in three steps. First, in Section 3.1, we derive the reservation value of an investor with utility type  $\delta$ , which allows us to characterize optimal trading rules and equilibrium asset prices given the joint distribution of utility types and asset holdings. Then, in Section 3.2, we use the trading rules to derive these joint distributions explicitly. Finally, in Section 3.3, we construct the unique equilibrium and show that it converges to a steady state from any initial allocation.

#### 3.1 Reservation values

Let  $V_{q,t}(\delta)$  denote the maximum attainable utility of an investor with  $q \in \{0, 1\}$  units of the asset and utility type  $\delta \in [0, 1]$  at time  $t \geq 0$ , and denote this investor’s *reservation value* by<sup>9</sup>

$$\Delta V_t(\delta) \equiv V_{1,t}(\delta) - V_{0,t}(\delta).$$

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<sup>8</sup>For simplicity, we will ignore throughout the paper the non-generic case where  $F(\delta)$  is flat at the level  $1 - s$  because, in such cases, the frictionless equilibrium price is not uniquely defined.

<sup>9</sup>Note that the reservation value function is well defined for all  $\delta \in [0, 1]$ , and not only for those utility types in the support of the underlying distribution,  $F(\cdot)$ .



In addition to considering an arbitrary distribution of utility types, our analysis of reservation values improves on the existing literature in several dimensions. First, in Section 3.1.1, we depart from the usual guess-and-verify approach by establishing elementary properties of reservation values directly, without making any *a priori* assumption on the direction of gains from trade. This allows us, down the road in Theorem 1, to claim a general uniqueness result for equilibrium. Second, in Section 3.1.2, we study a differential representation of reservation values which generalizes an earlier closed-form solution for the trading surplus in DGP's two-type model. Third, in Section 3.1.3, we study a sequential representation of reservation values which generalizes the concept of a marginal investor to an asset market with search-and-matching frictions.

### 3.1.1 Elementary properties

An application of Bellman's principle of optimality shows that

$$\begin{aligned}
V_{1,t}(\delta) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(u-t)} \delta du + e^{-r(\tau-t)} \left( \mathbf{1}_{\{\tau=\tau_1\}} V_{1,\tau}(\delta) \right. \right. \\
+ \mathbf{1}_{\{\tau=\tau_\gamma\}} \int_0^1 V_{1,\tau}(\delta') dF(\delta') \\
\left. \left. + \mathbf{1}_{\{\tau=\tau_0\}} \int_0^1 \max\{V_{1,\tau}(\delta), V_{0,\tau}(\delta) + P_\tau(\delta, \delta')\} \frac{d\Phi_{0,\tau}(\delta')}{1-s} \right) \right], \tag{3}
\end{aligned}$$

where  $\tau_\gamma$  is an exponential random variable with parameter  $\gamma$  that represents the arrival of a preference shock,  $\tau_q$  is an exponential random variable with parameter  $\lambda s$  if  $q = 1$  and  $\lambda(1-s)$  if  $q = 0$  that represents the occurrence of a meeting with a randomly selected investor who owns  $q$  units of the asset, the expectation is conditional on  $\tau \equiv \min\{\tau_0, \tau_1, \tau_\gamma\} > t$ , and

$$P_\tau(\delta, \delta') \equiv \theta_0 \Delta V_\tau(\delta) + \theta_1 \Delta V_\tau(\delta') \tag{4}$$

denotes the Nash solution to the bargaining problem at time  $\tau$  between an asset owner of utility type  $\delta$  and a non-owner of utility type  $\delta'$ . Substituting the price (4) into (3) and simplifying shows

that the maximum attainable utility of an asset owner satisfies

$$\begin{aligned}
V_{1,t}(\delta) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(u-t)} \delta du + e^{-r(\tau-t)} \left( V_{1,\tau}(\delta) \right. \right. \\
+ \mathbf{1}_{\{\tau=\tau_\gamma\}} \int_0^1 (V_{1,\tau}(\delta') - V_{1,\tau}(\delta)) dF(\delta') \\
\left. \left. + \mathbf{1}_{\{\tau=\tau_0\}} \int_0^1 \theta_1 (\Delta V_\tau(\delta') - \Delta V_\tau(\delta)) + \frac{d\Phi_{0,\tau}(\delta')}{1-s} \right) \right]. \quad (5)
\end{aligned}$$

The first term on the right-hand side of (3) accounts for the fact that an asset owner enjoys a constant flow of utility at rate  $\delta$  until time  $\tau$ . The remaining terms capture the three possible events for the asset owner at the stopping time  $\tau$ : he can receive a preference shock ( $\tau = \tau_\gamma$ ), in which case a new utility type is drawn from the distribution  $F(\delta')$ ; he can meet another asset owner ( $\tau = \tau_1$ ), in which case there are no gains from trade and his continuation payoff is  $V_{1,\tau}(\delta)$ ; or he can meet a non-owner ( $\tau = \tau_0$ ), who is of type  $\delta'$  with probability  $d\Phi_{0,\tau}(\delta')/(1-s)$ , in which case he sells the asset if the payoff from doing so exceeds the payoff from keeping the asset and continuing to search.

Proceeding in a similar way for  $q = 0$  shows that the maximum attainable utility of an investor who does not own an asset satisfies

$$\begin{aligned}
V_{0,t}(\delta) = \mathbb{E}_t \left[ e^{-r(\tau-t)} \left( V_{0,\tau}(\delta) + \mathbf{1}_{\{\tau=\tau_\gamma\}} \int_0^1 (V_{0,\tau}(\delta') - V_{0,\tau}(\delta)) dF(\delta') \right. \right. \\
\left. \left. + \mathbf{1}_{\{\tau=\tau_1\}} \int_0^1 \theta_0 (\Delta V_\tau(\delta) - \Delta V_\tau(\delta')) + \frac{d\Phi_{1,\tau}(\delta')}{s} \right) \right], \quad (6)
\end{aligned}$$

and subtracting (6) from (5) shows that the reservation value function satisfies the autonomous dynamic programming equation

$$\begin{aligned}
\Delta V_t(\delta) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(u-t)} \delta du + e^{-r(\tau-t)} \left( \Delta V_\tau(\delta) \right. \right. \\
+ \mathbf{1}_{\{\tau=\tau_\gamma\}} \int_0^1 (\Delta V_\tau(\delta') - \Delta V_\tau(\delta)) dF(\delta') \\
+ \mathbf{1}_{\{\tau=\tau_0\}} \int_0^1 \theta_1 (\Delta V_\tau(\delta') - \Delta V_\tau(\delta)) + \frac{d\Phi_{0,\tau}(\delta')}{1-s} \\
\left. \left. - \mathbf{1}_{\{\tau=\tau_1\}} \int_0^1 \theta_0 (\Delta V_\tau(\delta) - \Delta V_\tau(\delta')) + \frac{d\Phi_{1,\tau}(\delta')}{s} \right) \right]. \quad (7)
\end{aligned}$$

This equation reveals that an investor's reservation value is influenced by two distinct option values, which have opposing effects. On the one hand, an investor who owns an asset has the option to search and find a non-owner who will pay even more for the asset; as shown on the third line, this option increases her reservation value. On the other hand, an investor who does not own an asset has the option to search and find an owner who will sell at an even lower price; as shown on the fourth line, this option decreases her willingness to pay and, hence, her reservation value.

To guarantee the global optimality of the trading decisions induced by (5) and (6), we further require that the maximum attainable utilities of owners and non-owners, and hence the reservation values, satisfy the transversality conditions

$$\lim_{t \rightarrow \infty} e^{-rt} V_{q,t}(\delta) = \lim_{t \rightarrow \infty} e^{-rt} \Delta V_t(\delta) = 0, \quad (q, \delta) \in \{0, 1\} \times [0, 1]. \quad (8)$$

The next proposition establishes the existence, uniqueness, and some elementary properties of solutions to (5), (6), and (7) that satisfy (8).

**Proposition 1** *There exists a unique function  $\Delta V : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$  that satisfies (7) subject to (8). This function is uniformly bounded, absolutely continuous in  $(t, \delta) \in \mathbb{R}_+ \times [0, 1]$ , and strictly increasing in  $\delta \in [0, 1]$  with a uniformly bounded derivative with respect to type. Given  $\Delta V_t(\delta)$ , there are unique functions  $V_{0,t}(\delta)$  and  $V_{1,t}(\delta)$  that satisfy (5), (6), and (8).*

The fact that reservation values are strictly increasing in  $\delta$  implies that, when an asset owner of type  $\delta$  meets a non-owner of type  $\delta' > \delta$ , they will always agree to trade. Indeed, these two investors face the same distributions of future trading opportunities and preference shocks. Thus, the only relevant difference between them is the difference in utility flow enjoyed from the asset, which implies that the reservation value of an investor of type  $\delta'$  is strictly larger than that of an investor of type  $\delta < \delta'$ . The monotonicity property holds regardless of the distributions  $\Phi_{q,t}(\delta)$ , which investors take as given when calculating their optimal trading strategy. Moreover, as we establish below, this property greatly simplifies the derivation of closed-form solutions for both reservation values and the equilibrium distribution of asset holdings and utility types.

### 3.1.2 Differential representation

Integrating both sides of (7) with respect to the distribution of  $\tau$ , and using the fact that reservation values are strictly increasing in utility type, we obtain that the reservation value function satisfies

the integral equation

$$\begin{aligned} \Delta V_t(\delta) = \int_t^\infty e^{-(r+\gamma+\lambda)(u-t)} & \left( \delta + \lambda \Delta V_u(\delta) + \gamma \int_0^1 \Delta V_u(\delta') dF(\delta') \right. \\ & + \lambda \int_\delta^1 \theta_1 (\Delta V_u(\delta') - \Delta V_u(\delta)) d\Phi_{0,u}(\delta') \\ & \left. - \lambda \int_0^\delta \theta_0 (\Delta V_u(\delta) - \Delta V_u(\delta')) d\Phi_{1,u}(\delta') \right) du. \end{aligned} \quad (9)$$

In addition, since Proposition 1 establishes that the reservation value function is absolutely continuous in  $(t, \delta) \in \mathbb{R}_+ \times [0, 1]$  with a bounded derivative with respect to type, we know that

$$\Delta V_t(\delta) = \Delta V_t(0) + \int_0^\delta \sigma_t(\delta') d\delta' \quad (10)$$

for some nonnegative and uniformly bounded function  $\sigma_t(\delta)$  that is itself absolutely continuous in time for almost every  $\delta \in [0, 1]$ . We naturally interpret this function as a measure of the local surplus in the decentralized market, since the gains from trade between a seller of type  $\delta$  and a buyer of type  $\delta + d\delta$  are approximately given by  $\sigma_t(\delta)d\delta$ .

Substituting the representation (10) into (9), changing the order of integration, and differentiating both sides of the resulting equation with respect to  $t$  and  $\delta$  reveals that the local surplus satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$(r + \gamma + \lambda\theta_1(1 - s - \Phi_{0,t}(\delta)) + \lambda\theta_0\Phi_{1,t}(\delta)) \sigma_t(\delta) = 1 + \dot{\sigma}_t(\delta) \quad (11)$$

at almost every point of  $\mathbb{R}_+ \times [0, 1]$ .

The concept of local surplus and its HJB equation is the natural generalization of the concept of trading surplus in DGP to non-stationary environments with arbitrary distributions of utility types. To see this precisely, recall that DGP characterized an equilibrium in a special case of our model: in a steady-state with two utility types,  $\delta_\ell \leq \delta_h$ . In that setting, the measures  $1 - s - \Phi_0(\delta)$  and  $\Phi_1(\delta)$  are constant over  $[\delta_\ell, \delta_h]$  and correspond to the masses of buyers and sellers, respectively, denoted by  $\mu_{hn}$  and  $\mu_{\ell o}$  in DGP. Using this property, integrating both sides of (11), and restricting attention to the steady gives:

$$(r + \gamma + \lambda\theta_1\mu_{hn} + \lambda\theta_0\mu_{\ell o}) (\Delta V(\delta_h) - \Delta V(\delta_\ell)) = \delta_h - \delta_\ell,$$

which is the surplus formula of DGP.

Given (11) we can now derive a closed-form solution for reservation values. A calculation provided in the Appendix shows that, together with the requirements of boundedness and absolute continuity in time, equation (11) uniquely pins down the local surplus as

$$\sigma_t(\delta) = \int_t^\infty e^{-\int_t^u (r+\gamma+\lambda\theta_1(1-s-\Phi_{0,\xi}(\delta))+\lambda\theta_0\Phi_{1,\xi}(\delta))d\xi} du. \quad (12)$$

Combining this explicit solution for the local surplus with (9) and (10) allows us to derive the reservation value function in closed-form.

**Proposition 2** *For any distributions  $\Phi_{0,t}(\delta)$  and  $\Phi_{1,t}(\delta)$  satisfying (1) and (2), the unique solution to (7) and (8) is explicitly given by*

$$\begin{aligned} \Delta V_t(\delta) = \int_t^\infty e^{-r(u-t)} & \left( \delta - \int_0^\delta \sigma_u(\delta') (\gamma F(\delta') + \lambda\theta_0\Phi_{1,u}(\delta')) d\delta' \right. \\ & \left. + \int_\delta^1 \sigma_u(\delta') (\gamma(1 - F(\delta')) + \lambda\theta_1(1 - s - \Phi_{0,u}(\delta'))) d\delta' \right) du, \end{aligned} \quad (13)$$

where the local surplus  $\sigma_t(\delta)$  is defined by (12).

We close this sub-section with several intuitive comparative static results for reservation values.

**Corollary 1** *For any  $(t, \delta) \in \mathbb{R}_+ \times [0, 1]$ , the reservation value  $\Delta V_t(\delta)$  increases if an investor can bargain higher selling prices (larger  $\theta_1$ ), if he expects to have higher future valuations (a first-order stochastic dominance shift in  $F(\delta')$ ), or if he expects to trade with higher-valuation counterparts (a first-order stochastic dominance shift in the path of either  $\Phi_{0,t'}(\delta')$  or  $\Phi_{1,t'}(\delta')$ ).*

To complement these results, note that an increase in the search intensity,  $\lambda$ , can either increase or decrease reservation values. This is because of the two option values discussed above: an increase in  $\lambda$  increases an owner's option value of searching for a buyer who will pay a higher price, which drives the reservation value up, but it also increases a non-owner's option value of searching for a seller who will offer a lower price, which has the opposite effect. As we will see below in Section 4.2 the net effect is ambiguous and depends on all parameters of the model.

### 3.1.3 Sequential representation

Differentiating both sides of (9) with respect to time shows that the reservation value function can be characterized as the unique bounded and absolutely continuous solution to the HJB equation

$$\begin{aligned} r\Delta V_t(\delta) = & \delta + \Delta \dot{V}_t(\delta) + \gamma \int_0^1 (\Delta V_t(\delta') - \Delta V_t(\delta)) dF(\delta') \\ & + \lambda \int_\delta^1 \theta_1 (\Delta V_t(\delta') - \Delta V_t(\delta)) d\Phi_{0,t}(\delta') + \lambda \int_0^\delta \theta_0 (\Delta V_t(\delta') - \Delta V_t(\delta)) d\Phi_{1,t}(\delta'). \end{aligned} \quad (14)$$

The following proposition shows that the solution to this equation can be represented as the present value of utility flows from the asset to a hypothetical investor whose utility type process is adjusted to reflect the frictions present in the market.

**Proposition 3** *The reservation value function can be represented as*

$$\Delta V_t(\delta) = \mathbb{E}_{t,\delta} \left[ \int_t^\infty e^{-r(s-t)} \hat{\delta}_s ds \right], \quad (15)$$

where the market-valuation process,  $\hat{\delta}_t$ , is a pure jump Markov process on  $[0, 1]$  with infinitesimal generator defined by

$$\mathcal{A}_t[v](\delta) \equiv \int_0^1 (v(\delta') - v(\delta)) (\gamma dF(\delta') + \mathbf{1}_{\{\delta' > \delta\}} \lambda \theta_1 d\Phi_{0,t}(\delta') + \mathbf{1}_{\{\delta' \leq \delta\}} \lambda \theta_0 d\Phi_{1,t}(\delta'))$$

for any uniformly bounded function  $v : [0, 1] \rightarrow \mathbb{R}$ .

Representations such as (15) are standard in frictionless asset pricing, where private values are obtained as the present value of cash flows under a probability constructed from marginal rates of substitution. The emergence of such a representation in a decentralized market is, to the best of our knowledge, new to this paper and can be viewed as generalizing the concept of the marginal investor. Namely, in the frictionless benchmark, the market valuation is constant and equal to the utility flow of the marginal investor,  $\delta^*$ , since investors can trade instantly at price  $\delta^*/r$ . In a decentralized market, the market valuation differs from  $\delta^*$  for two reasons. First, because meetings are not instantaneous, an owner must enjoy his private utility flow until he finds a trading partner. Second, investors do not always trade with the marginal type. Instead, the terms of trade are random and depend on the distribution of types among trading partners. Importantly, this second channel is only active if there are more than two utility types, because otherwise a single price gets realized in bilateral meetings.

## 3.2 The joint distribution of asset holdings and types

In this section, we provide a closed-form characterization of the joint equilibrium distribution of asset holdings and utility types, in and out of steady state. To the best of our knowledge, this characterization is new to the literature. In particular, even in their special two-type case, DGP did not derive an explicit characterization of out-of-steady state dynamics. We then establish that this distribution converges to the steady-state from any initial conditions satisfying (1) and (2). Finally, we discuss several properties of a steady-state distribution and explain how its shape depends on the arrival rates of preference shocks and trading opportunities.

Since reservation values are increasing in utility type, trade occurs between two investors if and only if one is an owner with utility type  $\delta'$  and the other is a non-owner with utility type  $\delta'' \geq \delta'$ . Investors with the same utility type are indifferent between trading or not, but whether they trade is irrelevant since they effectively exchange ownership type. As a result, the rate of change in the measure of owners with utility type less than or equal to a given  $\delta \in [0, 1]$  satisfies

$$\dot{\Phi}_{1,t}(\delta) = \gamma (s - \Phi_{1,t}(\delta)) F(\delta) - \gamma \Phi_{1,t}(\delta) (1 - F(\delta)) - \lambda \Phi_{1,t}(\delta) (1 - s - \Phi_{0,t}(\delta)). \quad (16)$$

The first term in equation (16) is the inflow due to type-switching: at each instant, a measure  $\gamma (s - \Phi_{1,t}(\delta))$  of owners with utility type greater than  $\delta$  draw a new utility type, which is less than or equal to  $\delta$  with probability  $F(\delta)$ . A similar logic can be used to understand the second term, which is the outflow due to type-switching. The third term is the outflow due to trade. In particular, a measure  $(\lambda/2)\Phi_{1,t}(\delta)$  of investors who own the asset and have utility type less than  $\delta$  initiate contact with another investor, and with probability  $1 - s - \Phi_{0,t}(\delta)$  that investor is a non-owner with utility type greater than  $\delta$ , so that trade ensues. The same measure of trades occur when non-owners with utility type greater than  $\delta$  initiate trade with owners with utility type less than  $\delta$ , so that the sum equals the third term in (16).<sup>10</sup>

Using (1), we can rewrite (16) as a first-order ordinary differential equation for the measure of

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<sup>10</sup>Note that trading generates positive *gross* inflow into the set of owners with utility type less than  $\delta$ , but zero *net* inflow. Indeed, a gross inflow arises when a non-owner with utility type  $\delta' \leq \delta$  meets an owner with an even lower type  $\delta'' < \delta'$ . By trading, the previous owner of utility type  $\delta''$  leaves the set, but the new owner of utility type  $\delta'$  enters the same set, resulting in zero net inflow.

asset owners with utility type less or equal to  $\delta$ :

$$\dot{\Phi}_{1,t}(\delta) = -\lambda\Phi_{1,t}(\delta)^2 - \Phi_{1,t}(\delta) (\gamma + \lambda(1 - s - F(\delta))) + \gamma s F(\delta). \quad (17)$$

Importantly, this Riccati equation for  $\delta$  is independent from all  $\delta' \neq \delta$ , and holds without imposing any regularity conditions on the distribution of utility types—it works for continuous distributions, discrete distributions, or mixture of both.<sup>11</sup> Proposition 4 below provides an explicit expression for the unique solution to this equation and shows that it converges to a unique, globally stable steady state. To state the result, let

$$\Lambda(\delta) \equiv \sqrt{(1 - s + \gamma/\lambda - F(\delta))^2 + 4s(\gamma/\lambda)F(\delta)},$$

and denote by

$$\Phi_1(\delta) = F(\delta) - \Phi_0(\delta) \equiv -\frac{1}{2}(1 - s + \gamma/\lambda - F(\delta)) + \frac{1}{2}\Lambda(\delta) \quad (18)$$

the steady-state distribution of owners with utility type less than or equal to  $\delta$ , i.e., the unique, strictly positive solution to  $\dot{\Phi}_{1,t}(\delta) = 0$ .

**Proposition 4** *At any time  $t \geq 0$  the measure of asset owners with utility type less than or equal to  $\delta \in [0, 1]$  is explicitly given by*

$$\Phi_{1,t}(\delta) = \Phi_1(\delta) + \frac{(\Phi_{1,0}(\delta) - \Phi_1(\delta)) \Lambda(\delta)}{\Lambda(\delta) + (\Phi_{1,0}(\delta) - \Phi_1(\delta) + \Lambda(\delta)) (e^{\lambda\Lambda(\delta)t} - 1)} \quad (19)$$

*and converges pointwise monotonically to the steady-state measure  $\Phi_1(\delta)$  defined in (18) from any initial condition satisfying (1) and (2).*

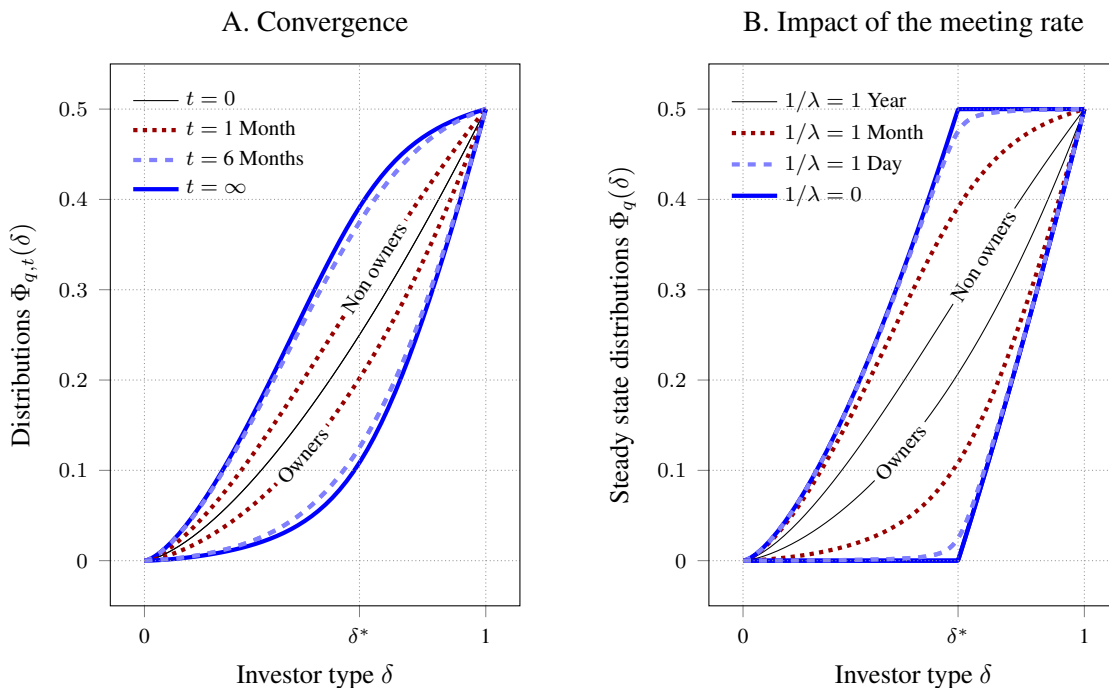
To illustrate the convergence of the equilibrium distributions to the steady state, we introduce a simple numerical example, which we will continue to use throughout the text. In this example, the discount rate is  $r = 0.05$ ; the asset supply is  $s = 0.5$ ; the meeting rate is  $\lambda = 12$ , so that a given investor meets others on average once a month; the arrival rate of preference shocks is  $\gamma = 1$ , so that investors change type on average once a year; the initial distribution of utility types

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<sup>11</sup>Alternatively, differentiating with respect to  $\delta$  reveals that the dynamic system for *measures* (instead of cumulative measures) does exhibit inter-dependence across values of  $\delta$ . That is, the equation characterizing the measure (or density) at  $\delta$  depends on the measures (or densities) at  $\delta' \neq \delta$ , making closed-form solutions more difficult to attain in all but the simplest cases.



FIGURE 1: Equilibrium distributions



Notes. The left panel plots the cumulative distribution of types among non-owners (upper curves) and owners (lower curves) at different points in time. The right panel plots these distributions in the steady state, for different levels of search frictions, indexed by the average inter-contact time,  $1/\lambda$ .

among asset owners is given by  $\Phi_{1,0}(\delta) = sF(\delta)$ ; and the underlying distribution of utility types is  $F(\delta) = \delta^\alpha$  with  $\alpha = 1.5$ , so that the marginal type is given by  $\delta^* = 0.6299$ .

Using this parameterization, the left panel of Figure 1 plots the equilibrium distributions among owners and non-owners at  $t = 0$ , after one month, after six months, and in the limiting steady state. As time passes, one can see that the assets are gradually allocated toward investors with higher valuations: the distribution of utility types among owners improves in the sense of first-order stochastic dominance (FOSD). Similarly, the distribution of utility types among non-owners deteriorates, in the FOSD sense, indicating that investors with low valuations are less and less likely to hold the asset over time.

Focusing on the steady-state distributions, (18) offers several natural comparative statics that we summarize in the following corollary.

**Corollary 2** For any  $\delta \in [0, 1]$ , the steady-state measure  $\Phi_1(\delta)$  of asset owners with utility type less than or equal to  $\delta$  is increasing in  $\gamma$  and decreasing in  $\lambda$ .

Intuitively, as preference shocks become less frequent (i.e.,  $\gamma$  decreases) or trading opportunities become more frequent (i.e.,  $\lambda$  increases), the asset is allocated to investors with higher valuations more efficiently, which implies an FOSD shift in the distribution of types among owners. In the limit, where types are permanent ( $\gamma \rightarrow 0$ ) or trading opportunities are constantly available ( $\lambda \rightarrow \infty$ ), the steady state distributions converge to their frictionless counterparts, as illustrated by the right panel of Figure 1, and the allocation is efficient. We return to this frictionless limit in Section 4.2, when we study the impact of search frictions on asset prices, misallocation, and welfare.

### 3.3 Equilibrium

**Definition 1** An equilibrium is a reservation value function  $\Delta V_t(\delta)$  and a pair of distributions  $\Phi_{0,t}(\delta)$  and  $\Phi_{1,t}(\delta)$  such that the distributions satisfy (1), (2) and (19), and the reservation value function satisfies (7) subject to (8) given the distributions.

Given the analysis above, a full characterization of the unique equilibrium is immediate. Note that uniqueness follows from the fact that we proved reservation values were strictly increasing directly, given arbitrary time paths for the distributions  $\Phi_{0,t}(\delta)$  and  $\Phi_{1,t}(\delta)$ , rather than guessing and verifying that such an equilibrium exists, as was done previously in the literature.

**Theorem 1** There exists a unique equilibrium. Moreover, given any initial conditions satisfying (1) and (2), this equilibrium converges to the steady state given by

$$\begin{aligned} r\Delta V(\delta) = & \delta - \int_0^\delta \sigma(\delta')(\gamma F(\delta') + \lambda\theta_0\Phi_1(\delta'))d\delta' \\ & + \int_\delta^1 \sigma(\delta')(\gamma(1 - F(\delta')) + \lambda\theta_1(1 - s - \Phi_0(\delta'))d\delta' \end{aligned} \quad (20)$$

with the time-invariant local surplus

$$\sigma(\delta) = \frac{1}{r + \gamma + \lambda\theta_1(1 - s - \Phi_0(\delta)) + \lambda\theta_0\Phi_1(\delta)},$$

and the steady-state cumulative distributions defined by (1) and (18).

## 4 Properties of Equilibrium

Given our characterization of the steady-state equilibrium above, it is possible to derive many implications of our model analytically. In this section, we focus on two—the distributions of *trading volume* and *prices* that arise in equilibrium—which highlight the importance of heterogeneity in pure decentralized asset markets. In particular, in Section 4.1, we study how investors with different valuations buy and sell assets at different rates, even though they meet other agents at the same rate. In the aggregate, we show that the heterogeneity in trading speed that arises in equilibrium generates endogenous intermediation, whereby a small fraction of investors account for a disproportionate amount of trading volume. Then, in Section 4.2, we show that heterogeneity magnifies the price impact of search frictions. Moreover, we show that this effect is more pronounced on price levels than it is on price dispersion and welfare.

### 4.1 Trading Patterns and the Concentration of Volume Across Investors

In this section, we first establish that, in our model, investors who have the most to gain from trading—i.e., those with extreme utility types and the “wrong” asset holdings—tend to find willing counterparties quickly. An immediate consequence of this seemingly elementary observation is that misallocation clusters around investors with utility types near the marginal type,  $\delta^*$ . Since these investors meet relatively frequently with both non-owners with higher utility types than their own and owners with lower utility types than their own, they find themselves intermediating a large fraction of the overall trading volume.

Therefore, even though the network of *meetings* generated by our model is random at any point in time, the network of *trades* is not. In particular, we show that this network has, endogenously, a *core-periphery* structure: over any time interval, if one created a connection between every pair of investors who trade, the network would exhibit what Jackson (2010, p. 67) describes as a “core of highly connected and interconnected nodes and a periphery of less-connected nodes.”<sup>12</sup> Hence, our model reveals an additional force that generates endogenous intermediation and a high

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<sup>12</sup>This type of trading network has been documented in many OTC markets, including the interdealer market for municipal bonds (Green, Hollifield, and Schürhoff, 2006; Li and Schürhoff, 2012), the interdealer market for securitization products (Hollifield, Neklyudov, and Spatt, 2014), the federal funds market (Bech and Atalay, 2010; Afonso and Lagos, 2012), the credit default swap market (Peltonen, Scheicher, and Vuillemeys, 2014), several foreign interbank markets (Craig and von Peter, 2014; Boss, Elsinger, Summer, and Thurner, 2004; Chang, Lima, Guerra,

concentration of trading volume in OTC markets, even when other dimensions of heterogeneity that are often used to explain core-periphery trading structures (such as capacity or contact rates) are not present.

#### 4.1.1 Trading intensity

The steady-state arrival rate of profitable trading opportunities for an owner with utility type  $\delta$ , or “selling intensity,” is the product of the arrival rate of a meeting and the probability that the investor meets a non-owner with utility type  $\delta' \geq \delta$ , i.e.,

$$\lambda_1(\delta) \equiv \lambda(1 - s - \Phi_0(\delta)).$$

Similarly, the steady-state arrival rate of profitable trading opportunities for a non-owner with utility type  $\delta$ , or “buying intensity,” is

$$\lambda_0(\delta) \equiv \lambda\Phi_1(\delta).$$

Since  $\Phi_0(\delta)$  is non-decreasing, the definition above implies that sellers with a higher utility type trade less often, and thus tend to hold the asset for longer periods. By the same logic, buyers with higher utility types trade more often, and thus tend to remain asset-less for shorter periods.

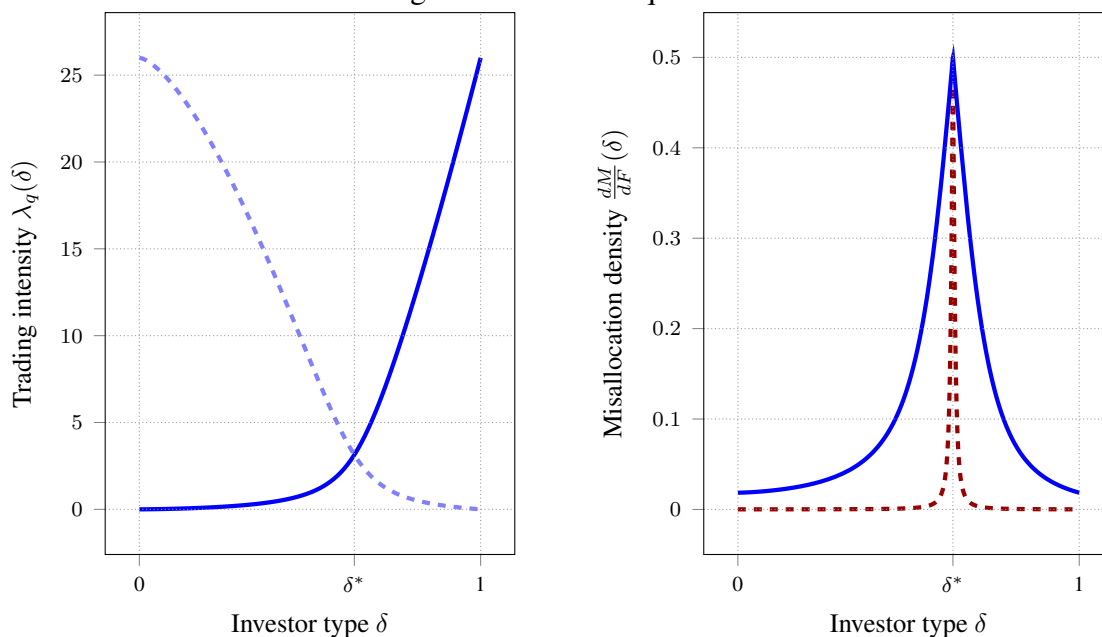
The left panel of Figure 2 uses the same parameterization of the economic environment as Figure 1 to plot the trading intensities  $\lambda_1(\delta)$  and  $\lambda_0(\delta)$  as functions of an investor’s utility type. In addition to confirming their monotonicity, the figure reveals that the trading intensities fall sharply for owners (non-owners) as their utility type approaches the marginal type  $\delta^*$  from below (above). Intuitively, for sufficiently large  $\lambda$ , the asset allocation becomes close to the frictionless allocation, *especially* at extreme utility types (see Panel B of Figure 1). Hence, owners with utility type  $\delta \gg \delta^*$  and non-owners with utility type  $\delta \ll \delta^*$  have essentially no willing counterparties to trade with, and thus their trading intensities are very low. The figure also reveals that the selling and buying intensities cross at the marginal type. Indeed,  $F(\delta^*) = 1 - s$  when the underlying distribution of utility types is continuous, and it follows that

$$\lambda_1(\delta^*) = \lambda(1 - s - \Phi_0(\delta^*)) = \lambda(F(\delta^*) - \Phi_0(\delta^*)) = \lambda\Phi_1(\delta^*) = \lambda_0(\delta^*).$$

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and Tabak, 2008), and even interbank flows across Fedwire, the large value transfer system operated by the Federal Reserve (Soramäki, Bech, Arnold, Glass, and Beyeler, 2007).

FIGURE 2: Trading intensities and equilibrium misallocation



Notes. The left panel plots the trading intensities of owners (dashed) and non owners (solid) when meetings happen on average once every month, while the right panel plots the misallocation density as functions of the investor's utility type when meetings happen on average once every month (solid) and once every hour (dashed). The parameters we use in this figure are otherwise the same as in Figure 1.

Hence, in equilibrium, buyers and sellers whose utility type are close to the marginal type tend to trade at the same speed.

The trading patterns described above illustrate that an investor's utility type endogenously determines his role in the market: those with extreme utility types trade infrequently and in the same direction, while those with moderate utility types (near  $\delta^*$ ) emerge as natural intermediaries, buying and selling more frequently and with approximately equal intensities. As we establish next, these trading patterns have important implications for the tendency of an investor to hold the wrong portfolio, relative to the frictionless benchmark.

### 4.1.2 Misallocation

We now study *misallocation*, defined as the extent to which the equilibrium asset allocation differs from its frictionless counterpart. To formalize this concept, let

$$M(\delta) = \int_0^\delta \mathbf{1}_{\{\delta' < \delta^*\}} d\Phi_1(\delta') + \int_0^\delta \mathbf{1}_{\{\delta' \geq \delta^*\}} d\Phi_0(\delta').$$

This measure is the sum of two types of misallocation: the measure of investors with utility type less than  $\delta$  who would own the asset in a frictionless environment, but do not own it in the presence of search frictions; and the measure of investors with utility type less than  $\delta$  who would not own the asset in a frictionless environment, but own it in the presence of search frictions.

To measure the extent of misallocation at a specific utility type, one can simply calculate the Radon-Nikodym density

$$\frac{dM}{dF} = \mathbf{1}_{\{\delta < \delta^*\}} \frac{d\Phi_1}{dF} + \mathbf{1}_{\{\delta \geq \delta^*\}} \frac{d\Phi_0}{dF} = \mathbf{1}_{\{\delta < \delta^*\}} \frac{d\Phi_1}{dF} + \mathbf{1}_{\{\delta \geq \delta^*\}} \left(1 - \frac{d\Phi_1}{dF}\right) \quad (21)$$

of the misallocation measure with respect to the measure induced by the underlying distribution of utility types; see equation (42) in the Appendix for an explicit expression. The value of the density  $\frac{dM}{dF}(\delta)$  represents the *fraction* of investors with utility type  $\delta$  whose holdings in the environment with search frictions differs from their holdings in the frictionless benchmark.

**Lemma 1** *The misallocation density  $\frac{dM}{dF}(\delta)$  achieves a global maximum at either  $\delta_-^*$  or  $\delta^*$ .*

The misallocation density has two key properties. First, it is non-monotonic and peaks at the marginal type,  $\delta^*$ . This arises because the selling intensity is decreasing in utility type, while the buying intensity is increasing. Second, as shown in the right panel of Figure 2, misallocation is highly concentrated near the marginal type. This occurs because there is an equilibrium feedback loop between the trading intensities and the distributions of utility types among owners and non-owners. For example, the non-monotonicity of the misallocation density means that there are relatively more non-owners at low utility types than near the marginal type. This implies that owners with low utility types are able to sell faster than those near the marginal type, which further reduces misallocation away from the marginal type, and increases misallocation near the marginal type. These reinforcing effects ultimately imply that misallocation is not only highest in a neighbourhood of the marginal type but tends to cluster around that point.

We emphasize that these two properties of misallocation arise in a decentralized market because trading intensities differ across utility types. Indeed, when all investors trade with equal intensity— as in frictionless models with centralized markets or in frictional models where all trades are executed by a set of dealers who have access to centralized markets —the measure of misallocation described above would be constant across utility types.

### 4.1.3 Trading volume across investors

Next, we show that the concentration of misallocation translates into a concentration of trading volume near the marginal type. To see this, let us first define trading volume as the flow rate of trades per unit time:

$$\vartheta = \lambda \int_{[0,1]^2} \mathbf{1}_{\{\delta_0 > \delta_1\}} d\Phi_0(\delta_0) d\Phi_1(\delta_1). \quad (22)$$

When the underlying distribution of utility types is continuous, we can use integration by parts to re-write equation (22) as

$$\begin{aligned} \vartheta = & \lambda \Phi_1(\delta^*) (1 - s - \Phi_0(\delta^*)) \\ & + \lambda \int_0^{\delta^*} dM(\delta) (\Phi_0(\delta^*) - \Phi_0(\delta)) + \lambda \int_{\delta^*}^1 dM(\delta) (\Phi_1(\delta) - \Phi_1(\delta^*)), \end{aligned}$$

with the misallocation measure defined in (21). The first term represents the volume generated by trades between owners with utility types in  $[0, \delta^*]$  and non-owners with utility types in  $[\delta^*, 1]$ ; these would be the only trades taking place in the equilibrium of a model with frictionless exchange. With search frictions, however, there are additional infra-marginal trades, captured by the second and third terms. In particular, the second term accounts for trades between owners with utility types  $\delta < \delta^*$  and non-owners with utility types in  $[\delta, \delta^*]$ , while the third term accounts for trades between non-owners with utility types  $\delta > \delta^*$  and owners with utility types in  $[\delta^*, \delta]$ .

The formula also highlights the role of misallocation in generating extra volume and suggests that near-marginal investors, who are characterized by greater misallocation, are likely to have a larger contribution to trading volume. This is confirmed in the next proposition.

**Proposition 5** *Assume that the distribution of utility types is continuous. Then the steady-state trading volume is explicitly given by*

$$\vartheta \equiv \gamma s (1 - s) \left[ (1 + \gamma/\lambda) \log \left( 1 + \frac{\lambda}{\gamma} \right) - 1 \right]. \quad (23)$$

In particular, the steady-state trading volume is strictly increasing in the meeting rate  $\lambda$ , with  $\lim_{\lambda \rightarrow \infty} \vartheta = \infty$  and

$$\lim_{\lambda \rightarrow \infty} \frac{\lambda}{\vartheta} \left( \int_{\delta^* - \varepsilon}^{\delta^*} \Phi_1(\delta) d\Phi_0(\delta) + \int_{\delta^*}^{\delta^* + \varepsilon} (1 - s - \Phi_0(\delta)) d\Phi_1(\delta) \right) = 1$$

for any constant  $\varepsilon > 0$  such that  $\delta^* \pm \varepsilon \in [0, 1]$ .

Proposition 5 establishes two key results. First, when the underlying distribution of utility types is continuous, the equilibrium trading volume is unbounded as  $\lambda \rightarrow \infty$ . By contrast, the equilibrium trading volume is finite in the frictionless benchmark.<sup>13</sup> Therefore, our fully decentralized market can generate arbitrarily large excess volume relative to the frictionless benchmark, as long as search frictions are sufficiently small.<sup>14</sup>

Second, trading volume is, for the most part, generated by investors near the marginal type who assume the role of intermediaries. To illustrate this phenomenon, Figure 3 plots the contribution of each owner-non-owner pair to the equilibrium trading volume, defined as

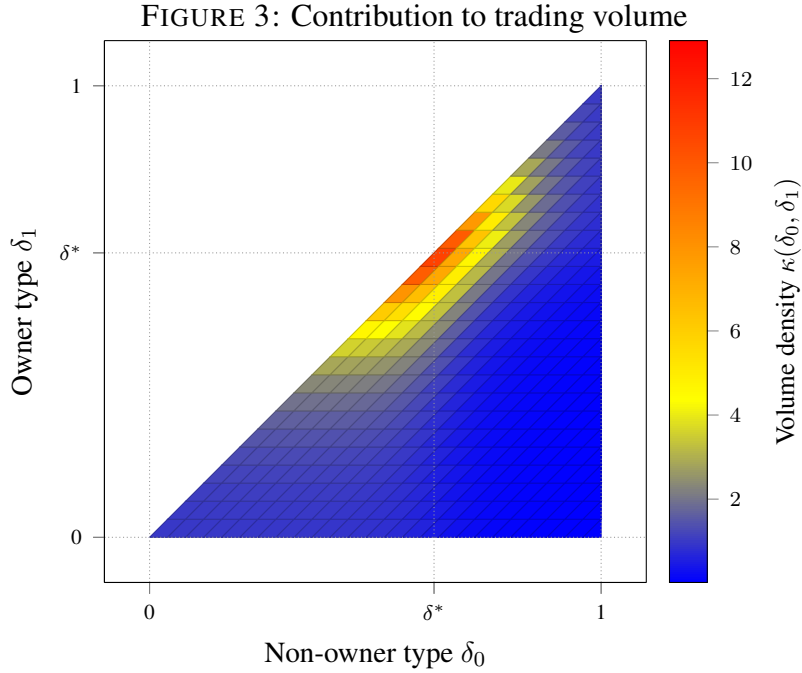
$$\kappa(\delta_0, \delta_1) = \mathbf{1}_{\{\delta_0 > \delta_1\}} \frac{d\Phi_0}{dF}(\delta_0) \frac{d\Phi_1}{dF}(\delta_1).$$

From the figure, one can see that investors with extreme utility types account for a small fraction of total trades and, therefore, lie at the periphery of the trading network. For example, owners with low utility types may trade quickly, but there are very few such owners in equilibrium. Hence, these owners contribute little to the trading volume. Likewise, there are many asset owners with high utility types, but these investors trade very slowly, so they do not account for many trades in equilibrium. Only in the cluster of investors with near-marginal utility types do we find a sufficiently large fraction of individuals who are both holding the “wrong” portfolio and able to meet suitable trading partners at a reasonably high rate—these are the investors that make up the core of the trading network.

<sup>13</sup>In a frictionless equilibrium, a measure  $s$  of agents hold the asset, all with type  $\delta > \delta^*$ . They sell as soon as they switch to a type  $\delta < \delta^*$ , which occurs with intensity  $\gamma(1 - F(\delta^*)) = \gamma(1 - s)$  by the market-clearing condition. Hence, the trading volume is equal to  $\gamma s(1 - s)$ .

<sup>14</sup>Equation (43) also delivers several additional comparative statics. For example, it shows that trading volume peaks when the asset supply equates the number of potential buyers and sellers—which is well-known from the monetary search literature (Kiyotaki and Wright, 1993)—and that it increases when investors change type more frequently.





Notes. This figure plots the volume density as a function of the owner's and non-owner's type when meetings occur, on average, once a week. The parameters we use in this figure are otherwise the same as in Figure 1.

## 4.2 The frictionless limit

We now study equilibrium prices and allocations as  $\lambda \rightarrow \infty$ . This is an important exercise for two reasons. First, this is the empirically relevant case in many financial markets, where trading speeds are becoming faster and faster. Second, as we establish below, this exercise highlights that heterogeneity magnifies the impact of search frictions on equilibrium outcomes.

As a first step, we establish two intuitive, but important, results about the economy as  $\lambda \rightarrow \infty$ : first, that the allocation converges to its frictionless counterpart; and second, that the reservation values of all investors converge to the frictionless equilibrium price,  $\delta^*/r$ .

**Proposition 6** *As search frictions vanish,  $\lim_{\lambda \rightarrow \infty} \Phi_0(\delta) = \Phi_0^*(\delta)$ ,  $\lim_{\lambda \rightarrow \infty} \Phi_1(\delta) = \Phi_1^*(\delta)$ , and  $\lim_{\lambda \rightarrow \infty} \Delta V(\delta) = \delta^*/r = p^*$  for every  $\delta \in [0, 1]$ .*

To understand why reservation values converge to the frictionless equilibrium price, consider the market-valuation process of Proposition 3. Since the equilibrium asset allocation becomes

approximately efficient as  $\lambda \rightarrow \infty$ , it becomes very easy for an investor with utility type  $\delta < \delta^*$  ( $\delta > \delta^*$ ) to sell (buy) an asset, but a lot more difficult to buy (sell) one. In particular, we show in Appendix A.2 that the trading intensities satisfy

$$\lim_{\lambda \rightarrow \infty} \lambda_0(\delta) \begin{cases} < \infty & \text{if } \delta < \delta^* \\ = \infty & \text{if } \delta > \delta^* \end{cases} \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \lambda_1(\delta) \begin{cases} = \infty & \text{if } \delta < \delta^* \\ < \infty & \text{if } \delta > \delta^* \end{cases} .$$

Thus, it follows from Proposition 3 that, starting from below (above) the marginal type, the market-valuation process moves up (down) very quickly as the meeting frequency increases. Taken together, these observations imply that the market-valuation process converges to  $\delta^*$  as  $\lambda \rightarrow \infty$ , and it now follows from the sequential representation (15) that all reservation values converge to the frictionless equilibrium price.

#### 4.2.1 Price level near the frictionless limit

To analyze the behavior of reservation values and prices near the frictionless limit, we study the behavior of the market-valuation process near the marginal type, which yields the following result.

**Proposition 7** *Assume that the distribution of utility types is twice continuously differentiable with a derivative that is bounded away from zero. Then,*

$$\Delta V(\delta) = p^* + \frac{\pi/r}{F'(\delta^*)} \left( \frac{1}{2} - \theta_0 \right) \left( \frac{\gamma s(1-s)}{\theta_0 \theta_1} \right)^{\frac{1}{2}} \frac{1}{\sqrt{\lambda}} + o\left( \frac{1}{\sqrt{\lambda}} \right), \quad (24)$$

for all utility types  $\delta \in [0, 1]$ .

The first term in the expansion follows directly from Proposition 6, since all reservation values converge to the frictionless price  $p^* = \delta^*/r$ . The main result of the proposition is the second term in the expansion, which determines the deviation of reservation values from the frictionless price. To calculate this term, we center the market-valuation process defined in Proposition 3 around its frictionless limit and scale it by its convergence rate, which turns out to be  $\sqrt{\lambda}$ . This delivers an auxiliary process  $\hat{x}_t = \sqrt{\lambda}(\hat{\delta}_t - \delta^*)$  whose limit distribution can be characterized explicitly, and the second term of the expansion is then obtained by calculating the limit of

$$\sqrt{\lambda}(\Delta V(\delta) - p^*) = \mathbb{E}_{\sqrt{\lambda}(\delta - \delta^*)} \left[ \int_0^\infty e^{-rt} \hat{x}_t dt \right].$$

We see from the proposition that the deviation from the frictionless price depends on three key features of our decentralized market model.

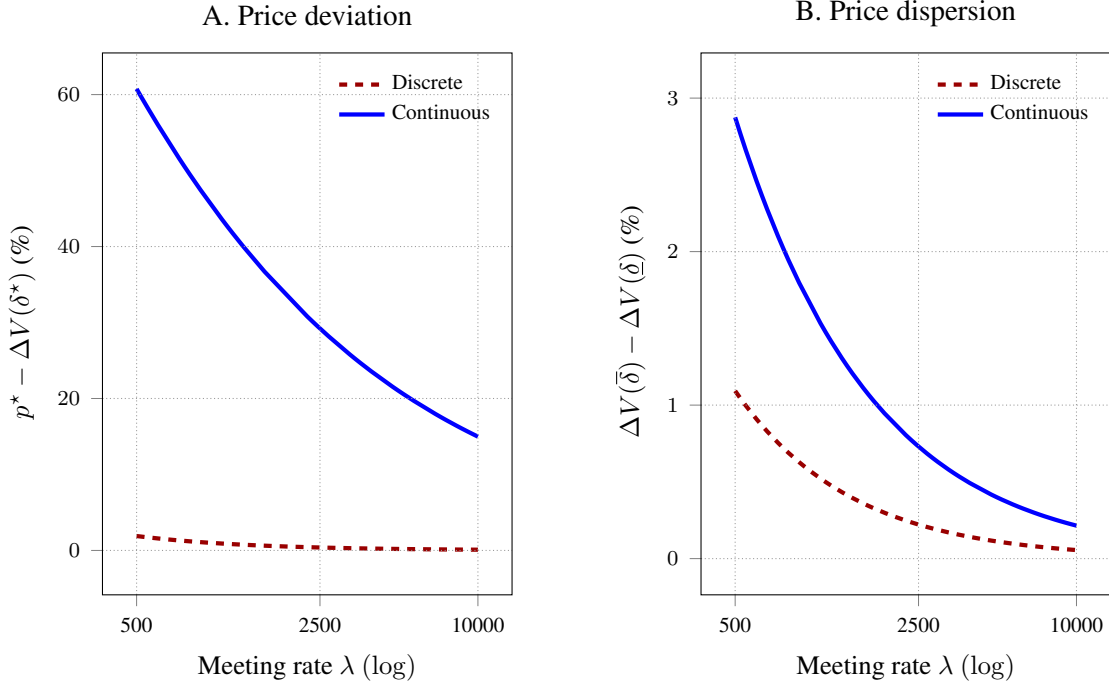
The first key feature is the average time it takes near-marginal investors to find counterparties, as measured by  $1/\sqrt{\lambda}$ . The second key feature is the relative bargaining powers of buyers and sellers, which determine whether the asset is traded at a discount or at a premium: if  $\theta_0 > 1/2$ , the asset is traded at a discount relative to the frictionless equilibrium price in all bilateral meetings, and vice versa if  $\theta_0 < 1/2$ . When buyers and sellers have equal bargaining powers, the correction term vanishes and all reservation values are well approximated by the frictionless price, irrespective of the other features of the market. The third feature of the market that matters for reservation values is the heterogeneity among investors in a neighborhood of the marginal type, as measured by the derivative  $F'(\delta^*)$  of the distribution at the marginal type. If the derivative is small, then valuations are dispersed around the marginal type, gains from trade are large, and bilateral bargaining induces significant deviations from the frictionless equilibrium price. On the contrary, if the derivative is large, then valuations are highly concentrated around the marginal type, gains from trade are small, and prices remain closer to their frictionless limit. Interestingly, a direct calculation shows that the derivative is proportional to the elasticity of the Walrasian demand at the frictionless price,

$$\varepsilon(p^*) = \frac{p^*}{F(rp^*) - 1} \frac{d(1 - F(rp))}{dp} \Big|_{p=p^*} = \delta^* F'(\delta^*)/s,$$

keeping in mind that  $1 - F(\delta^*) = s$ . Hence, holding the marginal investor and the supply the same, if the Walrasian demand is less elastic, price effects in the decentralized market will be larger. It is intuitive that a less elastic demand magnifies the bilateral monopoly effects at play in our search-and-matching market.

To further emphasize the role of heterogeneity, consider what happens when the continuous distribution of utility types approximates a discrete distribution. In such a case, the cumulative distribution function will approach a step function that is vertical at the marginal type, where demand is perfectly elastic. As a result, the derivative  $F'(\delta^*)$  will approach infinity, and it follows from (24) that the corresponding deviation from the frictionless equilibrium price will be very small. This informal argument can be made precise by working out the asymptotic expansion of reservation values with a discrete distribution of utility types.

FIGURE 4: Continuous vs. discrete distribution



Notes. This figures plots the price deviation relative to the frictionless equilibrium (left panel) and the price dispersion (right panel) as functions of the meeting rate for the base case model of Figure 1 with bargaining power  $\theta_0 = 0.75$ , and a model with a two point distribution of types constructed to have the same mean and to induce the same marginal investor as the continuous distribution of the base case model.

**Proposition 8** *When the distribution of utility types is discrete, the convergence rate of reservation values to the frictionless equilibrium price is generically equal to  $1/\lambda$ .*

To understand the different convergence rates in Propositions 7 and 8, consider a sequence of discrete distributions converging weakly to some continuous distribution. A simple argument shows that the corresponding allocations and prices converge to their continuous counterparts, but the asymptotic expansions of reservation values do not. Specifically, the proof of Proposition 8 reveals that, in the expansion with a discrete distribution, the coefficient multiplying  $1/\lambda$  diverges as the discrete distribution approaches its continuous limit. This means that convergence is slower and slower. Proposition 7 makes this observation mathematically precise by showing that, in the continuous limit, the convergence rate switches from  $1/\lambda$  to  $1/\sqrt{\lambda}$ .

To see that the difference in convergence rates is economically significant, let us compare the

price deviation  $p^* - \Delta V(\delta^*)$  implied by the continuous distribution of our baseline example with that implied by a two-point distribution, constructed to keep the marginal and average investors the same. The left panel of Figure 4 shows that, when investors meet counterparties twice a day on average (i.e.,  $\lambda = 500$ ), the deviation is 60 percent for the continuous distribution, and only about 2 percent for the corresponding discrete distribution. When meetings occur 20 times per day on average (i.e.,  $\lambda = 10,000$ ), the deviation is 15 percent for the continuous distribution, but it is now indistinguishable from zero for the discrete distribution. Why is there such a quantitatively large difference in price impact? According to our analysis, the difference is driven by a fundamental economic difference between the two classes of distributions: the elasticity of asset demand is infinite with a discrete distribution, and finite with a continuous one.

#### 4.2.2 Price dispersion near the frictionless limit

An important implication of Proposition 7 is that, to a first-order approximation, there is no price dispersion. This can be seen by noting that the correction term in (24) does not depend on the investor's utility type. Hence, in order to obtain results about the impact of frictions on price dispersion, it is necessary to work out higher order terms. This is the content of our next result.

**Proposition 9** *Assume that the distribution of utility types is twice continuously differentiable with a derivative that is bounded away from zero. Then*

$$\Delta V(1) - \Delta V(0) = \frac{1}{2\theta_0\theta_1 F'(\delta^*)} \frac{\log(\lambda)}{\lambda} + O\left(\frac{1}{\lambda}\right),$$

*By contrast, with a discrete distribution of utility types, the convergence rate of the price dispersion is generically equal to  $1/\lambda$ .*

Comparing the results of Propositions 7 and 9 shows that, with a continuous distribution of utility types, the price dispersion induced by search frictions vanishes at a rate  $\log(\lambda)/\lambda$ , which is much faster than the rate  $1/\sqrt{\lambda}$  at which reservation values converge to the frictionless equilibrium price. This finding has important implications for empirical analysis of decentralized markets, as it implies that inferring the impact of search frictions based on the observable level of price dispersion can be misleading. In particular, search frictions can have a very small impact on price dispersion and, yet, have a large impact on the equilibrium price level.

This finding is illustrated in Figure 4. Comparing the left and right panels, one sees clearly that the price dispersion induced by search frictions converges to zero much faster than the price deviation. For instance, when investors meet counterparties twice a day on average, the price discount implied by our baseline model is about 60 percent, but the corresponding price dispersion is about 20 times smaller. One can also see from the figure that, in accordance with the result of Proposition 9, price dispersion is larger with a continuous distribution of utility types than with a discrete distribution.

### 4.2.3 Welfare near the frictionless limit

In the analysis above, we established that the asymptotic behavior of two liquidity measures—the deviation of price from its frictionless limit and the dispersion of prices—provide quantitatively different signals about market liquidity. We now ask how these two measures are related to the welfare cost of frictions, defined as

$$C(\lambda) \equiv \int_{\delta^*}^1 \delta d\Phi_0(\delta) - \int_0^{\delta^*} \delta d\Phi_1(\delta).$$

In words,  $C(\lambda)$  is the difference between the collective flow utility of investors in the market with and without frictions: the first term accounts for the forgone utility of those investors who do not hold an asset in the frictional market when they should (according to the frictionless benchmark), while the second term accounts for the extra utility attributed to those investors who hold an asset in the frictional market when they should not.

**Proposition 10** *Assume that the distribution of utility types is twice continuously differentiable with a derivative that is bounded away from zero. Then*

$$C(\lambda) = \frac{\gamma s(1-s) \log(\lambda)}{F'(\delta^*) \lambda} + O\left(\frac{1}{\lambda}\right).$$

*By contrast, with a discrete distribution of utility types, the convergence rate of the welfare cost to zero is generically equal to  $1/\lambda$ .*

Proposition 10 establishes that search frictions have a larger welfare impact when the distribution of utility types is continuous than they do when the distribution is discrete—as was the case for price levels and price dispersion. The proposition also reveals that, as trading gets faster, the

welfare cost of frictions is accurately measured by the observed amount of price dispersion, since the two quantities converge to their frictionless counterparts at the same speed.

Of course, this does not necessarily imply that deviations in the price level do not matter; in a general equilibrium model, one would expect such deviations to distort the cost of capital and investment, which would have implications that are not captured by our model.

## **5 Conclusion**

We analyze a search and bargaining model of asset markets in which investors' valuations are drawn from an arbitrary distribution. The main contribution is methodological: we develop a solution technique that allows for a full characterization of the equilibrium, in closed form, both in and out of steady state. The result is a framework that is much richer than the popular workhorse model with only two valuations, yet equally tractable. As such, the model offers a variety of novel implications, and can be used to confront newer, transaction-level data emerging from OTC markets.

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# Appendix: Heterogeneity in Decentralized Asset Markets

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## A Proofs from Main Text

### A.1 Proofs omitted in Section 3

We start by showing that imposing the transversality condition (8) on the reservation value function is equivalent to seemingly stronger requirement of uniform boundedness, and that any such solution to the reservation value equation must be strictly increasing in utility types.

**Lemma A.1** *Any solution to (7) that satisfies (8) is uniformly bounded and strictly increasing in  $\delta \in [0, 1]$ .*

**Proof.** To facilitate the presentation we define the operator

$$\begin{aligned} \mathcal{O}_t[f](\delta) = \int_0^1 (f_t(\delta') - f_t(\delta)) (\gamma dF(\delta') + \lambda \theta_1 \mathbf{1}_{\{f_t(\delta') \geq f_t(\delta)\}} d\Phi_{0,t}(\delta') \\ + \lambda \theta_0 \mathbf{1}_{\{f_t(\delta') \leq f_t(\delta)\}} d\Phi_{1,t}(\delta')). \end{aligned}$$

Integrating with respect to the conditional distribution of the stopping time  $\tau$  shows that a solution to the reservation value equation (7) is a fixed point of the operator

$$T_t[f](\delta) = \int_t^\infty e^{-(r+\gamma+\lambda)(u-t)} (\delta + (\gamma + \lambda)f_u(\delta) + \mathcal{O}_u[f](\delta)) du. \quad (25)$$

Assume that  $\Delta V_t(\delta) = T_t[\Delta V](\delta)$  is a fixed point that satisfies (8). Since the right-hand side of (25) is absolutely continuous in time, we have that  $\Delta V_t(\delta)$  inherits this property, and it thus follows from Lebesgue's differentiation theorem that

$$\dot{\Delta} V_t(\delta) = r\Delta V_t(\delta) - \delta - \mathcal{O}_t[\Delta V](\delta)$$

for every  $\delta \in [0, 1]$  and almost every  $t \geq 0$ . Using this equation together with an integration by parts then

shows that the given solution satisfies

$$\Delta V_t(\delta) = e^{-r(H-t)} \Delta V_H(\delta) + \int_t^H e^{-r(u-t)} (\delta + \mathcal{O}_u[\Delta V](\delta)) du \quad (26)$$

$$= \lim_{H \rightarrow \infty} \int_t^H e^{-r(u-t)} (\delta + \mathcal{O}_u[\Delta V](\delta)) du \quad (27)$$

for all  $(\delta, t) \in \mathcal{S} \equiv \mathbb{R}_+ \times [0, 1]$  and any constant horizon  $t \leq H < \infty$  where the second equality follows from the transversality condition. Now assume towards a contradiction that the given solution fails to be nondecreasing in space so that  $\Delta V_t(\delta) > \Delta V_t(\delta')$  for some  $(t, \delta) \in \mathcal{S}$  and  $1 \geq \delta' > \delta$ . Because the right-hand side of (25) is absolutely continuous in time, this assumption implies that

$$H^* \equiv \inf \{u \geq t : \Delta V_u(\delta) \leq \Delta V_u(\delta')\} > t.$$

By definition we have that

$$\Delta V_u(\delta) \geq \Delta V_u(\delta'), \quad t \leq u \leq H^* \quad (28)$$

and, because the continuous functions  $x \mapsto (y-x)^+$  and  $x \mapsto -(x-y)^+$  are both non-increasing for every fixed  $y \in \mathbb{R}$ , it follows that

$$\mathcal{O}_u[\Delta V](\delta) \leq \mathcal{O}_u[\Delta V](\delta'), \quad t \leq u \leq H^*. \quad (29)$$

To proceed further, we distinguish two cases depending on whether the constant  $H^*$  is finite or not. Assume first that it is finite. In this case it follows from (26) that we have

$$\Delta V_t(\delta) = \int_t^{H^*} e^{-r(u-t)} (\delta + \mathcal{O}_u[\Delta V](\delta)) du + e^{-r(H^*-t)} \Delta V_{H^*}(\delta),$$

and combining this identity with (29) then gives

$$\begin{aligned} \Delta V_t(\delta) &\leq \int_t^{H^*} e^{-r(u-t)} (\delta + \mathcal{O}_u[\Delta V](\delta')) du + e^{-r(H^*-t)} \Delta V_{H^*}(\delta) \\ &= \int_t^{H^*} e^{-r(u-t)} (\delta + \mathcal{O}_u[\Delta V](\delta')) du + e^{-r(H^*-t)} \Delta V_{H^*}(\delta') < \Delta V_t(\delta'), \end{aligned} \quad (30)$$

where the equality follows by continuity, and the second inequality follows from the fact that  $\delta < \delta'$ . Now assume that  $H^* = \infty$  so that (28) and (29) hold for all  $u \geq t$ . In this case, (27) implies that

$$\Delta V_t(\delta) \leq \lim_{H \rightarrow \infty} \int_t^H e^{-r(u-t)} (\delta + \mathcal{O}_u[\Delta V](\delta')) du < \Delta V_t(\delta').$$

Combining this inequality with (30) delivers the required contradiction and establishes that  $\Delta V_t(\delta)$  is non-decreasing. To see that it is strictly increasing, rewrite (25) as

$$T_t[f](\delta) = \int_t^\infty e^{-\rho(u-t)} (\delta + \mathcal{M}_u[f](\delta)) du. \quad (31)$$

with the operator

$$\begin{aligned}\mathcal{M}_u[f](\delta) &= \lambda\eta f_u(\delta) + \gamma \int_0^1 f_u(\delta') dF(\delta') + \lambda\theta_0 \int_0^1 \min\{f_u(\delta'), f_u(\delta)\} d\Phi_{1,u}(\delta') \\ &\quad + \lambda\theta_1 \int_0^1 \max\{f_u(\delta'), f_u(\delta)\} d\Phi_{0,u}(\delta'),\end{aligned}$$

and the constants  $\rho \equiv r + \gamma + \lambda$  and  $\eta \equiv 1 - s\theta_0 - (1 - s)\theta_1$ . Because  $\mathcal{M}_u[f](\delta)$  is increasing in  $f_u(\delta)$  and the given solution is non-decreasing in space, we have that

$$\Delta V_t(\delta') - \Delta V_t(\delta) = \int_t^\infty e^{-\rho(u-t)} (\delta' - \delta + \mathcal{M}_u[\Delta V](\delta') - \mathcal{M}_u[\Delta V](\delta)) du \geq \frac{\delta' - \delta}{\rho}$$

for any  $0 \leq \delta \leq \delta' \leq 1$ , and the required strict monotonicity follows. To conclude the proof, it remains to establish boundedness. Because the given solution is increasing, we have

$$\sup_{t \geq 0} \mathcal{O}_t[\Delta V](1) \leq 0 \leq \inf_{t \geq 0} \mathcal{O}_t[\Delta V](0)$$

and it now follows from (27) that  $0 \leq \Delta V_t(0) \leq \Delta V_t(\delta) \leq \Delta V_t(1) \leq 1/r$  for all  $(t, \delta) \in \mathcal{S}$ . ■

**Proof of Proposition 1.** By Lemma A.1, we have that the existence, uniqueness, and strict (positive) monotonicity of a solution to (7) such that (8) holds is equivalent to the existence and uniqueness of a fixed point of the operator  $T$  in the space  $\mathcal{X}$  of uniformly bounded, measurable functions from  $\mathcal{S}$  to  $\mathbb{R}$  equipped with the sup norm. As is easily seen from (31) we have that  $T$  maps  $\mathcal{X}$  into itself. Moreover, using the definition of  $\mathcal{M}_u[f](\delta)$ , one easily sees that  $T$  satisfies the Blackwell's sufficient condition for a contraction (see Theorem 3.3 in Stokey and Lucas, 1989), with modulus  $\frac{\gamma + \lambda}{r + \gamma + \lambda}$ . The existence of a unique fixed point in the space  $\mathcal{X}$  now follows from the contraction mapping theorem because  $r > 0$  by assumption. To establish the second part, let  $\mathcal{X}_k$  denote the subset of functions  $f \in \mathcal{X}$  that are nonnegative and non-decreasing in space with

$$0 \leq f_t(\delta') - f_t(\delta) \leq \frac{\delta' - \delta}{r + \gamma} \equiv k(\delta' - \delta) \tag{32}$$

for all  $0 \leq \delta \leq \delta' \leq 1$  and  $t \geq 0$ . Let further  $\mathcal{X}_k^*$  denote the set of functions  $f \in \mathcal{X}_k$  that are strictly increasing in space and absolutely continuous with respect to time and space and observe that, because the set  $\mathcal{X}_k$  is closed in  $\mathcal{X}$ , it suffices to prove that  $T$  maps  $\mathcal{X}_k$  into  $\mathcal{X}_k^*$ . Fix an arbitrary  $f \in \mathcal{X}_k$ . Since this function is nonnegative, it follows from (31) that  $T_t[f](\delta)$  is nonnegative. On the other hand, using the inequalities in (32) in conjunction with the definition of the constant  $\eta$ , the increase of  $f_t(\delta)$  and the fact that the functions  $x \mapsto \min\{a; x\}$  and  $x \mapsto \max\{a; x\}$  are non-decreasing and Lipschitz continuous with constant one, we deduce that

$$0 \leq \mathcal{M}_t[f](\delta'') - \mathcal{M}_t[f](\delta) \leq \lambda k(\delta'' - \delta)$$

for all  $0 \leq \delta \leq \delta'' \leq 1$  and  $t \geq 0$ . Combining these inequalities with (31) and the definition of  $k$  then shows that we have

$$\frac{\delta'' - \delta}{r + \gamma + \lambda} \leq T_t[f](\delta'') - T_t[f](\delta) \leq \frac{(1 + \lambda k)(\delta'' - \delta)}{r + \gamma + \lambda} = k(\delta'' - \delta)$$

for all  $0 \leq \delta \leq \delta'' \leq 1$  and  $t \geq 0$ . Taken together, these bounds imply that the function  $T_t[f](\delta)$  is strictly increasing in space and belongs to  $\mathcal{X}_k$ , so it now only remains to establish absolute continuity. By definition of the set  $\mathcal{X}_k$  we have that

$$f_t(\delta) = f_t(\delta') + \int_{\delta'}^{\delta} \phi_t(x) dx$$

for all  $t \geq 0$ , almost every  $\delta, \delta' \in [0, 1]^2$ , and some  $0 \leq \phi_t(x) \leq k$ . Substituting this identity into (25) and changing the order of integration shows that

$$\begin{aligned} T_t[f](\delta) = \int_t^{\infty} e^{-(r+\gamma+\lambda)(u-t)} & \left( \delta + (\lambda + \gamma)f_u(\delta) - \int_0^{\delta} \phi_u(\delta')(\gamma F(\delta') + \lambda\theta_0\Phi_{1,u}(\delta')) d\delta' \right. \\ & \left. + \int_{\delta}^1 \phi_u(\delta')(\gamma(1 - F(\delta')) + \lambda\theta_1(1 - s - \Phi_{0,u}(\delta'))) d\delta' \right) du \end{aligned} \quad (33)$$

and the required absolute continuity now follows from [Sremr \(2010, Theorem 3.1\)](#). ■

**Lemma A.2** *Given the reservation value function there exists a unique pair of functions  $V_{1,t}(\delta)$  and  $V_{0,t}(\delta)$  that satisfy (3) and (6) subject to (8).*

**Proof.** Assume that  $V_{1,t}(\delta)$  and  $V_{0,t}(\delta)$  satisfy (3) and (6) subject to (8). Integrating on both sides of (3) and (6) with respect to the conditional distribution of the stopping time  $\tau$  shows that

$$V_{q,t}(\delta) = \int_t^{\infty} e^{-(r+\gamma+\lambda)(u-t)} (\lambda V_{q,u}(\delta) + \mathcal{C}_{q,u}(\delta) + \gamma \int_0^1 V_{q,u}(\delta') dF(\delta')) du. \quad (34)$$

with the uniformly bounded functions defined by

$$\mathcal{C}_{q,t}(\delta) = q\delta + \int_0^1 \lambda\theta_q ((2q - 1)(\Delta V_t(\delta') - \Delta V_t(\delta)))^+ d\Phi_{1-q,t}(\delta'). \quad (35)$$

Because the right-hand side of (34) is absolutely continuous in time, we have that the functions  $V_{q,t}(\delta)$  inherit this property, and it thus follows from Lebesgue's differentiation theorem that

$$\dot{V}_{q,t}(\delta) = rV_{q,t}(\delta) - \mathcal{C}_{q,t}(\delta) - \gamma \int_0^1 (V_{q,t}(\delta') - V_{q,t}(\delta)) dF(\delta') \quad (36)$$

for all  $\delta \in [0, 1]$  and almost every  $t \geq 0$ . Combining this differential equation with the assumed transversality condition then implies that

$$\begin{aligned} V_{q,t}(\delta) &= e^{-r(H-t)} V_{q,H}(\delta) + \int_t^H e^{-r(u-t)} (\mathcal{C}_{q,u}(\delta) + \gamma \int_0^1 (V_{q,u}(\delta') - V_{q,u}(\delta)) dF(\delta')) du \\ &= \lim_{H \rightarrow \infty} \int_t^H e^{-r(u-t)} (\mathcal{C}_{q,u}(\delta) + \gamma \int_0^1 (V_{q,u}(\delta') - V_{q,u}(\delta)) dF(\delta')) du \end{aligned}$$

for any finite horizon and, because the functions  $\mathcal{C}_{q,t}(\delta)$  are increasing in space by [Lemma A.4](#) below, the same arguments as in the proof of [Lemma A.1](#) show that the functions  $V_{q,t}(\delta)$  are increasing in space and

are uniformly bounded. Combining these properties with (36) then shows that the process

$$e^{-rt}V_{q,t}(\delta_t) + \int_0^t e^{-ru}\mathcal{C}_{q,u}(\delta_u)du$$

is a uniformly bounded martingale in the filtration generated by the investor's utility type process, and it follows that we have

$$V_{q,t}(\delta) = \mathbb{E}_{t,\delta} \left[ \int_t^\infty e^{-r(u-t)}\mathcal{C}_{q,u}(\delta_u)du \right]. \quad (37)$$

This establishes the uniqueness of the solutions to (3) and (6) subject to (8) and it now only remains to show that these solutions are consistent with the given reservation value function. Applying the law of iterated expectations to (37) at the stopping time  $\tau$  shows that the function  $V_{1,t}(\delta) - V_{0,t}(\delta)$  is a uniformly bounded fixed point of the operator

$$U_t[f](\delta) = \int_t^\infty e^{-(r+\gamma+\lambda)(u-t)}(\lambda f_u(\delta) + \mathcal{C}_{1,u}(\delta) - \mathcal{C}_{0,u}(\delta) + \gamma \int_0^1 f_u(\delta')dF(\delta'))du.$$

A direct calculation shows that this operator is a contraction on  $\mathcal{X}$  and, therefore, admits a unique fixed point in  $\mathcal{X}$ . Because the reservation value function is increasing we have

$$\mathcal{C}_{1,t}(\delta) - \mathcal{C}_{0,t}(\delta) + \gamma \int_0^1 \Delta V_t(\delta')dF(\delta') = \delta + \gamma \Delta V_t(\delta) + \mathcal{O}_t[\Delta V](\delta)$$

and it follows that this fixed point coincides with the reservation value function. ■

**Lemma A.3** *For any fixed  $\delta \in [0, 1]$ , the unique solution to (11) that is both absolutely continuous in time and uniformly bounded is explicitly given by*

$$\sigma_t(\delta) = \int_t^\infty e^{-\int_t^u R_\xi(\delta)d\xi}du, \quad (38)$$

with the effective discount rate  $R_t(\delta) = r + \gamma + \lambda\theta_1(1 - s - \Phi_{0,t}(\delta)) + \lambda\theta_0\Phi_{1,t}(\delta)$ .

**Proof.** Fix an arbitrary  $\delta \in [0, 1]$  and assume that  $\sigma_t(\delta)$  is a uniformly bounded solution to (11) that is absolutely continuous in time. Using integration by parts, we easily obtain that

$$\sigma_t(\delta) = e^{-\int_t^T R_\xi(\delta)d\xi}\sigma_T(\delta) + \int_t^T e^{-\int_t^u R_\xi(\delta)d\xi}du, \quad 0 \leq t \leq T < \infty.$$

Since  $\sigma \in \mathcal{X}$  and  $R_t(\delta) \geq \gamma > 0$ , we have that

$$\lim_{T \rightarrow \infty} e^{-\int_t^T R_\xi(\delta)d\xi}\sigma_T(\delta) = 0$$

and therefore

$$\sigma_t(\delta) = \lim_{T \rightarrow \infty} \left( e^{-\int_t^T R_\xi(\delta)d\xi}\sigma_T(\delta) + \int_t^T e^{-\int_t^u R_\xi(\delta)d\xi}du \right) = \int_t^\infty e^{-\int_t^s R_u(\delta)du}ds$$

by monotone convergence. ■

**Lemma A.4** *The functions  $C_{q,t}(\delta)$  are increasing in  $\delta \in [0, 1]$ .*

**Proof.** For  $q = 0$  the result follows immediately from (35) and the fact that the reservation value function is increasing in  $\delta \in [0, 1]$ . Assume now that  $q = 1$ . Using the fact that the reservation value function is increasing and integrating by parts on the right-hand side of equation (35) gives

$$C_{1,t}(\delta) = \delta + \int_{\delta}^1 \lambda \theta_1 \sigma_t(\delta') (1 - s - \Phi_{1,t}(\delta')) d\delta',$$

and differentiating this expression shows that

$$C'_{1,t}(\delta) = 1 - \lambda \sigma_t(\delta) \theta_1 (1 - s - \Phi_{1,t}(\delta)) \geq 1 - \frac{\lambda \theta_1 (1 - s)}{r + \gamma + \lambda(\theta_0 s + \theta_1 (1 - s))} > 0,$$

where the first inequality follows from (38) and the definition of  $R_t(\delta)$ , and the last inequality follows from the strict positive of the interest rate. ■

**Proof of Proposition 2.** Let the local surplus  $\sigma_t(\delta)$  be as above and consider the absolutely continuous function defined by

$$f_t(\delta) = \int_t^{\infty} e^{-r(u-t)} \left( \delta - \int_0^{\delta} \sigma_u(\delta') (\gamma F(\delta') + \lambda \theta_0 \Phi_{1,u}(\delta')) d\delta' \right. \\ \left. + \int_{\delta}^1 \sigma_u(\delta') (\gamma(1 - F(\delta')) + \lambda \theta_1 (1 - s - \Phi_{0,u}(\delta'))) d\delta' \right) du.$$

Using the uniform boundedness of the functions  $\sigma_t(\delta)$ ,  $F(\delta)$ , and  $\Phi_{q,t}(\delta)$ , we deduce that  $f \in \mathcal{X}$ . On the other hand, Lebesgue's differentiation theorem implies that this function is almost everywhere differentiable in both the time and the space variable with

$$\dot{f}_t(\delta) = r f_t(\delta) - \delta + \int_0^{\delta} \sigma_t(\delta') (\gamma F(\delta') + \lambda \theta_0 \Phi_{1,t}(\delta')) d\delta' \\ - \int_{\delta}^1 \sigma_t(\delta') (\gamma(1 - F(\delta')) + \lambda \theta_1 (1 - s - \Phi_{0,t}(\delta'))) d\delta' \quad (39)$$

for all  $\delta \in [0, 1]$  and almost every  $t \geq 0$ , and

$$f'_t(\delta) = \int_t^{\infty} e^{-r(u-t)} (1 - \sigma_u(\delta) (\gamma + \lambda \theta_1 (1 - s - \Phi_{0,u}(\delta)) + \lambda \theta_0 \Phi_{1,u}(\delta))) du \\ = \int_t^{\infty} e^{-r(u-t)} (r \sigma_u(\delta) - \dot{\sigma}_u(\delta)) du = \sigma_t(\delta)$$

for all  $t \geq 0$  and almost every  $\delta \in [0, 1]$ , where the second equality follows from (11) and the third follows from integration by parts and the boundedness of the local surplus. In particular, the fundamental theorem of calculus implies

$$f_t(\delta') - f_t(\delta) = \int_{\delta}^{\delta'} \sigma_t(\delta'') d\delta'', \quad (\delta, \delta') \in [0, 1]^2, \quad (40)$$

and it follows that  $f_t(\delta)$  is strictly increasing in space. Using this monotonicity in conjunction with (40) and integrating by parts on the right-hand side of (39) shows that

$$\dot{f}_t(\delta) = r f_t(\delta) - \delta - \mathcal{O}_t[f](\delta)$$

for all  $\delta \in [0, 1]$  and almost every  $t \geq 0$ . Writing this differential equation as

$$(r + \gamma + \lambda) f_t(\delta) - \dot{f}_t(\delta) = \delta + (\gamma + \lambda) f_t(\delta) + \mathcal{O}_t[f](\delta)$$

and integrating by parts then shows that

$$f_t(\delta) = e^{-(r+\gamma+\lambda)(H-t)} f_H(\delta) + \int_t^H e^{-(r+\gamma+\lambda)(u-t)} (\delta + (\gamma + \lambda) f_u(\delta) + \mathcal{O}_u[f](\delta)) du$$

for any  $t \leq H < \infty$ , and it now follows from the dominated convergence theorem and the uniform boundedness of the function  $f_t(\delta)$  that

$$f_t(\delta) = \int_t^\infty e^{-(r+\gamma+\lambda)(u-t)} (\delta + (\gamma + \lambda) f_u(\delta) + \mathcal{O}_u[f](\delta)) du.$$

Comparing this expression with (25), we conclude that  $f_t(\delta) = T_t[f](\delta) \in \mathcal{X}$ , and the desired result now follows from the uniqueness established in the proof of Proposition 1. ■

**Proof of Corollary 1.** As shown in the proof of Proposition 1, we have that  $\Delta V_t(\delta)$  is the unique fixed point of the contraction  $T : \mathcal{X}_k \rightarrow \mathcal{X}_k$  defined by (31) and, by inspection, this mapping is increasing in  $f_t(\delta)$  and decreasing in  $r$ . Furthermore, it follows from equation (33) that  $T$  is increasing in  $\theta_1$  and decreasing in  $\theta_0$ ,  $F(\delta)$  and  $\Phi_{q,t}(\delta)$  and the desired monotonicity now follows from Lemma A.5 below. ■

**Lemma A.5** *Let  $\mathcal{C} \subseteq \mathcal{X}$  be closed and assume that  $A[\cdot; \alpha] : \mathcal{C} \rightarrow \mathcal{C}$  is a contraction that is increasing in  $f$  and increasing (resp. decreasing) in  $\alpha$ . Then its fixed point is increasing (resp. decreasing) in  $\alpha$ .*

**Proof.** Assume that  $A_t[f; \alpha](\delta)$  is a contraction on  $\mathcal{C} \subset \mathcal{X}$  that is increasing in  $(\alpha, f)$  and denote its fixed point by  $f_t(\delta; \alpha)$ . Combining the assumed monotonicity with the fixed-point property shows that

$$f_t(\delta; \alpha) = A_t[f(\cdot; \alpha); \alpha](\delta) \leq A_t[f(\cdot; \alpha); \beta](\delta), \quad (t, \delta) \in \mathcal{S}.$$

Iterating this relation gives

$$f_t(\delta; \alpha) \leq A_t^n[f; \beta](\delta), \quad (t, \delta, n) \in \mathcal{S} \times \{1, 2, \dots\}$$

and the desired result now follows by taking limits on both sides as  $n \rightarrow \infty$  and using the fact that the mapping  $A[\cdot; \beta]$  is a contraction. ■

**Proof of Proposition 3.** Using (14) together with the notation of the statement shows that the reservation value function is the unique bounded and absolutely continuous solution to

$$r \Delta V_t(\delta) = \dot{\Delta V}_t(\delta) + \delta + \mathcal{A}_t[\Delta V](\delta).$$



Therefore, it follows from an application of Itô's lemma that the process

$$e^{-rt} \Delta V_t(\hat{\delta}_t) + \int_0^t e^{-ru} \hat{\delta}_u du$$

is a local martingale, and this implies that we have

$$\Delta V_t(\delta) = \mathbb{E}_{t,\delta} \left[ e^{-r(\tau_n-t)} \Delta V_{\tau_n}(\hat{\delta}_{\tau_n}) \right] + \mathbb{E}_{t,\delta} \left[ \int_0^{\tau_n} e^{-r(u-t)} \hat{\delta}_u du \right]$$

for a non-decreasing sequence of stopping times that converges to infinity. Since the reservation value function is uniformly bounded, we have that the first term on the right-hand side converges to zero as  $n \rightarrow \infty$ , and the desired result now follows by monotone convergence. ■

**Proof of Proposition 4.** For a fixed  $\delta \in [0, 1]$ , the differential equation

$$-\dot{\Phi}_{1,t}(\delta) = \lambda \Phi_{1,t}(\delta)^2 + \lambda \Phi_{1,t}(\delta)(1 - s + \gamma/\lambda - F(\delta)) - \gamma s F(\delta)$$

is a Riccati equation with constant coefficients whose unique solution can be found in any textbook on ordinary differential equations; see, for example, Reid (1972). Let us now turn to the convergence part. Using (1) and (2) together with the definition of  $\Lambda(\delta)$  and  $\Phi_q(\delta)$  shows that the term

$$\begin{aligned} \Phi_{1,0}(\delta) - \Phi_1(\delta) + \Lambda(\delta) &= \Phi_{1,0}(\delta) + \frac{1}{2}(1 - s + \gamma/\lambda - F(\delta) + \Lambda(\delta)) \\ &= \Phi_{1,0}(\delta) + \Phi_1(\delta) + (1 - s + \gamma/\lambda - F(\delta)) = \gamma/\lambda + \Phi_1(\delta) + (1 - s - \Phi_{0,0}(\delta)) \end{aligned}$$

that appears in the denominator of (19) is nonnegative for all  $\delta \in [0, 1]$ . Since  $\lambda \Lambda(\delta) > 0$ , this implies that the nonnegative function

$$|\Phi_{1,t}(\delta) - \Phi_1(\delta)| = \frac{|\Phi_{1,0}(\delta) - \Phi_1(\delta)| \Lambda(\delta)}{\Lambda(\delta) + (\Phi_{1,0}(\delta) - \Phi_1(\delta) + \Lambda(\delta))(e^{\lambda \Lambda(\delta)t} - 1)}$$

is monotone decreasing in time and converges to zero as  $t \rightarrow \infty$ . ■

**Lemma A.6** *The steady-state cumulative distribution of types among owners  $\Phi_1(\delta)$  is increasing in the asset supply, and increasing and concave in  $\phi = \gamma/\lambda$ , with*

$$\begin{aligned} \lim_{\phi \rightarrow 0} \Phi_1(\delta) &= sF(\delta), \text{ and} \\ \lim_{\phi \rightarrow \infty} \Phi_1(\delta) &= (F(\delta) - 1 + s)^+. \end{aligned}$$

*In particular, the steady-state cumulative distributions functions  $\Phi_q(\delta)$  converge to their frictionless counterparts as  $\lambda \rightarrow \infty$ .*

**Proof of Lemma A.6.** A direct calculation shows that

$$\frac{\partial \Phi_1(\delta)}{\partial s} = \frac{\Phi_1(\delta) + \phi F(\delta)}{\Lambda(\delta)},$$

and the desired monotonicity in  $s$  follows. On the other hand, using the definition of the steady-state distribution, it can be shown that

$$\frac{\partial \Phi_1(\delta)}{\partial \phi} = \frac{sF(\delta) - \Phi_1(\delta)}{\Lambda(\delta)} = \frac{s(1-s)F(\delta)(1-F(\delta))}{(\phi + \Phi_1(\delta) + (1-s)(1-F(\delta)))\Lambda(\delta)} \quad (41)$$

and the desired monotonicity follows by observing that all the terms on the right-hand side are nonnegative. Knowing that  $\Phi_1(\delta)$  is increasing in  $\phi$ , we deduce that

$$\Lambda(\delta) = 2\Phi_1(\delta) + 1 - s + \phi - F(\delta)$$

is also increasing in  $\phi$ , and it now follows from the first equality in (41) that

$$\frac{\partial^2 \Phi_1(\delta)}{\partial \phi^2} = -\frac{1}{\Lambda(\delta)} \frac{\partial \Phi_1(\delta)}{\partial \phi} \left( 1 + \frac{\partial \Lambda(\delta)}{\partial \phi} \right) \leq 0.$$

The expressions for the limiting values follow by sending  $\phi$  to zero and  $\infty$  in the definition of the steady-state distribution. ■

**Proof of Corollary 2.** The result follows directly from Lemma A.6. ■

**Proof of Theorem 1.** The result follows directly from the definition, Proposition 1, and Proposition 4. We omit the details. ■

## A.2 Proofs omitted in Section 4

To simplify the notation, let  $\phi \equiv \gamma/\lambda$ . The following lemma follows immediately from the equation defining the steady-state distribution of utility types among asset owners.

**Lemma A.7** *The steady-state distributions of types satisfy  $\Phi_1(\delta) = F(\delta) - \Phi_0(\delta) = \ell(F(\delta))$ , where the bounded function*

$$\ell(x) \equiv -\frac{1}{2}(1-s+\phi-x) + \frac{1}{2}\sqrt{(1-s+\phi-x)^2 + 4s\phi x}$$

*is the unique positive solution to  $\ell^2 + (1-s+\phi-x)\ell - s\phi x = 0$ . Moreover, the function  $\ell(x)$  is strictly increasing and convex, and strictly so if  $s \in (0, 1)$ .*

**Proof of Lemma A.7.** It is obvious that  $\ell(x)$  is the unique positive solution of the second-order polynomial shown above. The function  $\ell(x)$  is strictly increasing by an application of the implicit function theorem: when  $x > 0$ ,  $\ell(x) > 0$ , so that the second-order polynomial must be strictly increasing in  $\ell$  and strictly decreasing in  $x$ . Convexity follows from the fact that

$$\ell''(x) = \frac{2s(1-s)\phi(1+\phi)}{\sqrt[3]{4s\phi x + (1-s+\phi-x)^2}} \geq 0,$$

with a strict inequality if  $s \in (0, 1)$ . ■

**Proof of Lemma 1.** Let  $\mathbf{A}$  denote the set of atoms of the distribution  $F(\delta)$ . With this definition, we have that the Radon-Nikodym density is given by

$$m(\delta) = \frac{dM}{dF}(\delta) \equiv \begin{cases} \mathbf{1}_{\{\delta \notin \mathbf{A}\}} \ell'(F(\delta)) + \mathbf{1}_{\{\delta \in \mathbf{A}\}} \frac{\Delta \ell(F(\delta))}{\Delta F(\delta)}, & \text{if } \delta < \delta^*, \\ \mathbf{1}_{\{\delta \notin \mathbf{A}\}} (1 - \ell'(F(\delta))) + \mathbf{1}_{\{\delta \in \mathbf{A}\}} \left(1 - \frac{\Delta \ell(F(\delta))}{\Delta F(\delta)}\right), & \text{otherwise.} \end{cases} \quad (42)$$

To establish the result, we need to show that  $m(\delta)$  is increasing on  $[0, \delta^*)$  and decreasing on  $[\delta^*, 1]$ . As shown in the proof of Lemma A.7, we have that the function  $\ell(x)$  is strictly convex on  $[0, 1]$ . This immediately implies that the functions  $\ell'(x)$  and  $(\ell(x) - \ell(y))/(x - y)$  are, respectively, increasing in  $x \in [0, 1]$  and increasing in  $x \in [0, 1]$  and  $y \in [0, 1]$ , and the desired result now follows from (42). ■

Assuming that a meeting between a buyer and a seller with the same utility results in trade with some constant probability  $\pi \in [0, 1]$  we can express the steady state trading volume as

$$\vartheta(\pi) = \lambda \int_{[0,1]^2} \mathbf{1}_{\{\delta_0 > \delta_1\}} d\Phi_0(\delta_0) d\Phi_1(\delta_1) + \pi \lambda \sum_{\delta \in [0,1]} \Delta \Phi_0(\delta) \Delta \Phi_1(\delta),$$

where  $\Delta \Phi_q(\delta) = \Phi_q(\delta) - \Phi_q(\delta_-) \geq 0$  denotes the discrete mass of investors who hold  $q \in \{0, 1\}$  units of the asset and have a utility type exactly equal to  $\delta$ .

**Lemma A.8** *If the distribution of utility types is continuous then*

$$\vartheta(\pi) = \vartheta_c \equiv \gamma s(1-s) \left[ \left(1 + \gamma/\lambda\right) \log \left(1 + \frac{\lambda}{\gamma}\right) - 1 \right] \quad (43)$$

for all  $\pi \in [0, 1]$  and is strictly increasing in both the meeting rate  $\lambda$  and the arrival rate of preference shocks  $\gamma$ . Otherwise, if the distribution of utility types has atoms, then the steady-state trading volume is strictly increasing in  $\pi \in [0, 1]$  with  $\vartheta(0) < \vartheta_c < \vartheta(1)$ .

**Proof of Lemma A.8.** Consider the continuous functions defined by

$$G_1(x) = \frac{\ell(x)}{s} \quad \text{and} \quad G_0(x) = \frac{x - \ell(x)}{1 - s}.$$

Rearranging the quadratic equation for  $\ell(x)$  given in Lemma A.7, it can be shown that these functions satisfy the identity

$$G_1(x) = \frac{\phi G_0(x)}{1 + \phi - G_0(x)}, \quad (44)$$

where  $\phi = \gamma/\lambda$ . Since the functions  $G_q(x)$  are continuous, strictly increasing, and map  $[0, 1]$  onto itself, we have that they each admit a continuous and strictly increasing inverse  $G_q^{-1}(y)$ , and it follows that identity (44) can be written equivalently as

$$G_1(G_0^{-1}(y)) = \frac{\phi y}{1 + \phi - y}. \quad (45)$$

Consider the class of tie-breaking rules whereby a fraction  $\pi \in [0, 1]$  of the meetings between an owner and a non-owner of the same utility type lead to a trade. By definition, the trading volume associated with such

a tie breaking rule can be computed as

$$\vartheta(\pi) = \lambda s(1-s) (\mathbb{P}[\delta_0 > \delta_1] + \pi \mathbb{P}[\delta_0 = \delta_1]),$$

where the random variables  $(\delta_0, \delta_1) \in [0, 1]^2$  are distributed according to  $\Phi_0(\delta)/(1-s) = G_0(F(\delta))$  and  $\Phi_1(\delta)/s = G_1(F(\delta))$  independently of each other. A direct calculation shows that the quantile functions of these random variables are given by

$$\inf\{x \in [0, 1] : G_q(F(x)) \geq u\} = \inf\{x \in [0, 1] : F(x) \geq G_q^{-1}(u)\} = \Delta(G_q^{-1}(u))$$

where  $\Delta(y)$  denotes the quantile function of the underlying distribution of utility types, and it thus follows from Lemma A.9 below that the trading volume satisfies

$$\frac{\vartheta(\pi)}{\lambda s(1-s)} = \mathbb{P}[\Delta(G_0^{-1}(u_0)) > \Delta(G_1^{-1}(u_1))] + \pi \mathbb{P}[\Delta(G_0^{-1}(u_0)) = \Delta(G_1^{-1}(u_1))],$$

where  $u_0$  and  $u_1$  denote a pair of iid uniform random variables. If the distribution is continuous, then its quantile function is strictly increasing, and the above identity simplifies to

$$\begin{aligned} \frac{\vartheta(\pi)}{\lambda s(1-s)} &= \mathbb{P}[G_0^{-1}(u_0) > G_1^{-1}(u_1)] = \mathbb{P}[u_1 < G_1(G_0^{-1}(u_0))] \\ &= \mathbb{E}[G_1(G_0^{-1}(u_0))] = \int_0^1 G_1(G_0^{-1}(x)) dx = \int_0^1 \frac{\phi x}{1 + \phi - x} dx = \frac{\vartheta^*}{\lambda s(1-s)}, \end{aligned}$$

where we used formula (45) for  $G_1(G_0^{-1}(y))$ , and the last equality follows from the calculation of the integral. If the distribution fails to be continuous, then its quantile function will have flat spots that correspond to the levels across which the distribution jumps, but it will nonetheless be weakly increasing. As a result, we have the strict inclusions

$$\{\Delta(G_0^{-1}(u_0)) > \Delta(G_1^{-1}(u_1))\} \subset \{G_0^{-1}(u_0) > G_1^{-1}(u_1)\} \subset \{\Delta(G_0^{-1}(u_0)) \geq \Delta(G_1^{-1}(u_1))\},$$

and it follows that

$$\begin{aligned} \frac{\vartheta(0)}{\lambda s(1-s)} &= \mathbb{P}[\Delta(G_0^{-1}(u_0)) > \Delta(G_1^{-1}(u_1))] \\ &< \mathbb{P}[G_0^{-1}(u_0) > G_1^{-1}(u_1)] = \frac{\vartheta^*}{\lambda s(1-s)} \\ &= \mathbb{P}[G_0^{-1}(u_0) \geq G_1^{-1}(u_1)] < \mathbb{P}[\Delta(G_0^{-1}(u_0)) \geq \Delta(G_1^{-1}(u_1))] = \frac{\vartheta(1)}{\lambda s(1-s)}. \end{aligned}$$

Since the function  $\vartheta(\pi)$  is continuous and strictly increasing in  $\pi$ , this further implies that there exists a unique tie-breaking probability  $\pi^*$  such that  $\vartheta^* = \vartheta(\pi^*)$  and the proof is complete.  $\blacksquare$

**Lemma A.9** *Let  $H(x)$  be a cumulative probability distribution function on  $[0, 1]$ . If the random variable  $U$  is uniformly distributed on  $[0, 1]$ , then the random variable  $\inf\{x \in [0, 1] : H(x) \geq U\}$  is distributed according to  $H(x)$ .*

**Proof.** Let  $\mathcal{X}(q) \equiv \{x' \in [0, 1] : H(x') \geq q\}$  and  $X(q) \equiv \inf \mathcal{X}(q)$ . We show that  $X(q) \leq x$  if and only if  $H(x) \geq q$ . For the ‘‘if’’ part, suppose that  $H(x) \geq q$ , then  $x$  belongs to  $\mathcal{X}(q)$  and is therefore larger

than its infimum, which is  $X(q) \leq x$ . For the “only if” part, let  $(x_n)_{n=1}^\infty \subseteq \mathcal{X}(q)$  be a decreasing sequence converging toward  $X(q)$ . For each  $n$ , we have that  $H(x_n) \geq q$ . Going to the limit and using the fact that  $H(x)$  is right continuous, we obtain that  $H(X(q)) \geq q$ , which implies  $H(x) \geq q$  since  $H(x)$  is increasing and  $X(q) \leq x$  for all  $x \in \mathcal{X}(q)$ . ■

**Proof of Proposition 5.** The first part of the result follows directly from Lemma A.8. To establish the second part let  $\varepsilon$  be as in the statement and assume that the distribution of utility types is continuous. In this case the equilibrium trading volume can be decomposed as

$$\begin{aligned} \vartheta_c = & \lambda \Phi_1(\delta^*)(1 - s - \Phi_0(\delta^*)) + \lambda \int_0^{\delta^* - \varepsilon} \Phi_1(\delta) d\Phi_0(\delta) + \lambda \int_{\delta^* + \varepsilon}^1 (1 - s - \Phi_0(\delta)) d\Phi_1(\delta) \\ & + \lambda \int_{\delta^* - \varepsilon}^{\delta^*} \Phi_1(\delta) d\Phi_0(\delta) + \lambda \int_{\delta^*}^{\delta^* + \varepsilon} (1 - s - \Phi_0(\delta)) d\Phi_1(\delta). \end{aligned} \quad (46)$$

We show that all the terms on the first line remain bounded as  $\lambda \rightarrow \infty$ . Since  $F(\delta^*) = 1 - s$  when the distribution of type is continuous we have that the first term is equal to

$$\lambda \Phi_1(\delta^*) (1 - s - F(\delta^*) + \Phi_1(\delta^*)) = \lambda \Phi_1(\delta^*)^2.$$

and we know from Lemma A.7 that the measure  $\Phi_1(\delta^*)$  of owners below the marginal type solves

$$\lambda \Phi_1(\delta^*)^2 + \gamma \Phi_1(\delta^*) - \gamma s(1 - s) = 0.$$

This immediately implies that

$$\lambda \Phi_1(\delta^*)^2 \leq \gamma s(1 - s)$$

and it follows that the first term on the first line of (46) remains bounded as  $\lambda \rightarrow \infty$ . Turning to the second term, we note that

$$\lambda \int_0^{\delta^* - \varepsilon} \Phi_1(\delta) d\Phi_0(\delta) \leq \lambda \Phi_1(\delta^* - \varepsilon) F(\delta^* - \varepsilon), \quad (47)$$

where the inequality follows (1) and the increases of  $\Phi_1(\delta)$ . From Lemma A.7, we have that the steady-state measure of owners with valuations below  $\delta^* - \varepsilon$  solves

$$\lambda \Phi_1(\delta^* - \varepsilon)^2 + \left(1 - s - F(\delta^* - \varepsilon) + \frac{\gamma}{\lambda}\right) \lambda \Phi_1(\delta^* - \varepsilon) - \gamma s F(\delta^* - \varepsilon) = 0.$$

This immediately implies that

$$\lambda \Phi_1(\delta^* - \varepsilon) \leq \frac{\gamma s F(\delta^* - \varepsilon)}{1 - s - F(\delta^* - \varepsilon)}.$$

and combining this inequality with (47) shows that the second term on the first line of (46) remains bounded as  $\lambda \rightarrow \infty$ . Proceeding similarly, one can show that the third term also remains bounded as frictions vanish, and the desired result now follows by observing that  $\lim_{\lambda \rightarrow \infty} \vartheta_c = \infty$ . ■

**Proof of Proposition 6.** From equation (18), it follows that we have

$$\lim_{\lambda \rightarrow \infty} \Phi_1(\delta) = \frac{|1 - s - F(\delta)|}{2} - \frac{1 - s - F(\delta)}{2} = (1 - s - F(\delta))^- = \Phi_1^*(\delta)$$

and therefore  $\lim_{\lambda \rightarrow \infty} \Phi_0(\delta) = \Phi_0^*(\delta)$  for all  $\delta \in [0, 1]$ . By Theorem 1, we have that the steady state reservation value function is explicitly given by

$$r\Delta V(\delta) = \delta - \int_0^\delta k_0(\delta') d\delta' + \int_\delta^1 k_1(\delta') d\delta'$$

with the uniformly bounded functions defined by

$$k_0(\delta') = \frac{\gamma F(\delta') + \lambda \theta_0 \Phi_1(\delta')}{r + \gamma + \lambda \theta_1 (1 - s - \Phi_0(\delta')) + \lambda \theta_0 \Phi_1(\delta')}$$

$$k_1(\delta') = \frac{\gamma(1 - F(\delta')) + \lambda \theta_1 (1 - s - \Phi_0(\delta'))}{r + \gamma + \lambda \theta_1 (1 - s - \Phi_0(\delta')) + \lambda \theta_0 \Phi_1(\delta')}.$$

Using the first part of the proof and the assumption that  $\theta_q > 0$ , we obtain

$$\lim_{\lambda \rightarrow \infty} k_q(\delta') = \frac{\theta_q \Phi_{1-q}^*(\delta')}{\theta_0 \Phi_1^*(\delta') + \theta_1 \Phi_0^*(\delta')} = \mathbf{1}_{\{q=0\}} \mathbf{1}_{\{\delta \geq \delta^*\}} + \mathbf{1}_{\{q=1\}} \mathbf{1}_{\{\delta < \delta^*\}},$$

and the required result now follows from an application of the dominated convergence theorem because the functions  $k_q(\delta')$  take values in  $[0, 1]$ . ■

**Convergence rates of the distributions.** To derive the rates at which the equilibrium distributions converge to their frictionless counterparts, recall the inflow-outflow equation that characterizes the steady-state equilibrium distributions:

$$\gamma F(\delta) (s - \Phi_1(\delta)) = \gamma \Phi_1(\delta) (1 - F(\delta)) + \lambda \Phi_1(\delta) (1 - s - \Phi_0(\delta)). \quad (48)$$

By Proposition 6 we have that  $\Phi_1(\delta) \rightarrow 0$  and  $\Phi_0(\delta) \rightarrow F(\delta) < 1 - s$  for all utility types  $\delta < \delta^*$  as the meeting frequency becomes infinite, and it thus follows from (48) that for  $\delta < \delta^*$  the distribution of utility types among asset owners admits the approximation

$$\Phi_1(\delta) = \frac{\gamma F(\delta) s}{1 - s - F(\delta)} \left( \frac{1}{\lambda} \right) + o\left( \frac{1}{\lambda} \right). \quad (49)$$

Similarly, by Proposition 6 we have that  $\Phi_1(\delta) \rightarrow F(\delta) - 1 + s > 0$  and  $\Phi_0(\delta) \rightarrow 1 - s$  for all utility types  $\delta > \delta^*$  as the meeting frequency becomes infinite, and it thus follows from (48) that for  $\delta > \delta^*$  the distribution of utility types among non-owners admits the approximation

$$1 - s - \Phi_0(\delta) = \frac{\gamma(1 - s)(1 - F(\delta))}{F(\delta) - (1 - s)} \left( \frac{1}{\lambda} \right) + o\left( \frac{1}{\lambda} \right). \quad (50)$$

To derive the convergence rate at the point  $\delta = \delta^*$ , assume first that the distribution of utility types crosses

the level  $1 - s$  continuously and observe that in this case we have

$$1 - s - \Phi_0(\delta^*) = 1 - s - F(\delta^*) + \Phi_1(\delta^*) = \Phi_1(\delta^*).$$

Substituting these identities into (48) evaluated at the marginal type and letting  $\lambda \rightarrow \infty$  on both sides shows that the equilibrium distributions admit the approximation given by

$$\Phi_1(\delta^*) = 1 - s - \Phi_0(\delta^*) = \sqrt{\gamma s(1-s)} \left( \frac{1}{\sqrt{\lambda}} \right) + o \left( \frac{1}{\sqrt{\lambda}} \right). \quad (51)$$

If the distribution of utility types crosses  $1 - s$  by a jump, we have  $F(\delta^*) > 1 - s$ , and it follows that the approximation (50) also holds at the marginal type. ■

**Proof of Proposition 7.** Assume without loss of generality that the support of the distribution of utility types is the interval  $[0, 1]$ . Evaluating (20) at  $\delta^*$  and making the change of variable  $x = \sqrt{\lambda}(\delta' - \delta^*)$  in the two integrals shows that

$$r\sqrt{\lambda}(\Delta V(\delta^*) - p^*) = P(\lambda) - D(\lambda), \quad (52)$$

where the functions on the right-hand side are defined by

$$D(\lambda) \equiv \int_{-\infty}^0 \mathbf{1}_{\{x+\delta^*\sqrt{\lambda} \geq 0\}} \frac{\gamma F(\delta^* + x/\sqrt{\lambda}) + \theta_0 \sqrt{\lambda} g_1(x)}{r + \gamma + \theta_0 \sqrt{\lambda} g_1(x) + \theta_1 \sqrt{\lambda} g_0(x)} dx$$

and

$$P(\lambda) \equiv \int_0^{\infty} \mathbf{1}_{\{x+\delta^*\sqrt{\lambda} \leq 1\}} \frac{\gamma(1 - F(\delta^* + x/\sqrt{\lambda})) + \theta_1 \sqrt{\lambda} g_0(x)}{r + \gamma + \theta_0 \sqrt{\lambda} g_1(x) + \theta_1 \sqrt{\lambda} g_0(x)} dx$$

with the functions

$$g_q(x) \equiv \frac{\lambda_{1-q}(\delta^* + x/\sqrt{\lambda})}{\sqrt{\lambda}} = \sqrt{\lambda}(1-q)(1-s - F(\delta^* + x/\sqrt{\lambda})) + \sqrt{\lambda}\Phi_1(\delta^* + x/\sqrt{\lambda}).$$

Letting the meeting rate  $\lambda \rightarrow \infty$  on both sides of equation (52) and using the convergence result established by Lemma A.12 below we obtain that

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} r\sqrt{\lambda}(\Delta V(\delta^*) - p^*) &= \int_0^{\infty} \frac{\theta_1 g(-x) dx}{\theta_0 g(x) + \theta_1 g(-x)} - \int_{-\infty}^0 \frac{\theta_0 g(z) dz}{\theta_0 g(z) + \theta_1 g(-z)} \\ &= \int_0^{\infty} \frac{(1 - 2\theta_0)g(x)g(-x) dx}{(\theta_0 g(x) + \theta_1 g(-x))(\theta_0 g(-x) + \theta_1 g(x))} dx \\ &= \int_0^{\infty} \frac{\gamma s(1-s)(1 - 2\theta_0) dx}{\gamma s(1-s) + \theta_0 \theta_1 (xF'(\delta^*))^2} = \frac{\pi}{F'(\delta^*)} \left( \frac{1}{2} - \theta_0 \right) \left( \frac{\gamma s(1-s)}{\theta_0 \theta_1} \right)^{\frac{1}{2}}, \end{aligned}$$

where the function

$$g(x) = \frac{1}{2} x F'(\delta^*) + \frac{1}{2} \sqrt{(xF'(\delta^*))^2 + 4\gamma s(1-s)}$$

is the unique positive solution to (53), the second equality follows by making the change of variable  $-z = x$  in the second integral, the third equality follows from the definition of the function  $g(x)$ , and the last equality follows from the fact that

$$\int_0^\infty \frac{dx}{a+x^2} = \left[ \frac{\arctan(x/\sqrt{a})}{\sqrt{a}} \right]_0^\infty = \frac{\pi}{2\sqrt{a}}, \quad a > 0.$$

This shows that the asymptotic expansion of the statement holds at the marginal type and the desired result now follows from the fact that  $\Delta V(\delta) = \Delta V(\delta^*) + o(1/\sqrt{\lambda})$  by Proposition 9. ■

**Lemma A.10** *Assume that the conditions of Proposition 7 hold and denote by  $g(x)$  the positive solution to the quadratic equation*

$$g^2 - gF'(\delta^*)x - \gamma s(1-s) = 0. \quad (53)$$

*Then we have that  $g_1(x) \rightarrow g(x)$  and  $g_0(x) \rightarrow g(-x)$  for all  $x \in \mathbb{R}$  as  $\lambda \rightarrow \infty$ .*

**Proof.** Evaluating (17) at the steady-state shows that the function  $g_1(x)$  is the unique positive solution to the quadratic equation given by

$$g^2 + \left[ \frac{\gamma}{\sqrt{\lambda}} + \sqrt{\lambda} \left( F(\delta^*) - F(\delta^* + x/\sqrt{\lambda}) \right) \right] g - \gamma s F(\delta^* + x/\sqrt{\lambda}) = 0. \quad (54)$$

Because the left hand side of this quadratic equation is negative at the origin and positive at  $g = 1$  we have that  $g_1(x) \in [0, 1]$ . This implies that  $g_1(x)$  has a well-defined limit as  $\lambda \rightarrow \infty$ , and it now follows from (54) that this limit is given by the positive solution to (53). Next, we note that

$$g_0(x) = g_1(x) + \sqrt{\lambda} \left( F(\delta^*) - F(\delta^* + x/\sqrt{\lambda}) \right).$$

Substituting this expression into equation (54) then shows that the function  $g_0(x)$  is the unique positive solution to the quadratic equation given by

$$g^2 + \left[ \frac{\gamma}{\sqrt{\lambda}} - \sqrt{\lambda} \left( F(\delta^*) - F(\delta^* + x/\sqrt{\lambda}) \right) \right] g - \gamma(1-s) \left( 1 - F(\delta^* + x/\sqrt{\lambda}) \right),$$

and the desired result follows from the same arguments as above. ■

**Lemma A.11** *Assume that the conditions of Proposition 7 hold. Then*

(a) *There exists a finite  $K \geq 0$  such that*

$$\begin{aligned} g_1(x) &\leq K/|x|, & x \in I_-^\lambda &\equiv [-\delta^*\sqrt{\lambda}, 0], \\ g_0(x) &\leq K/|x|, & x \in I_+^\lambda &\equiv [0, (1-\delta^*)\sqrt{\lambda}]. \end{aligned} \quad (55)$$

(b) *For any given  $\bar{x} \in I_+^\lambda \cap (-I_-^\lambda)$ , there exists a strictly positive  $k$  such that*

$$\begin{aligned} g_1(x) &\geq k|x|, & x \in I_+^\lambda \cap [\bar{x}, \infty), \\ g_0(x) &\geq k|x|, & x \in I_-^\lambda \cap (-\infty, -\bar{x}] \end{aligned} \quad (56)$$



for all sufficiently large  $\lambda$ .

**Proof.** Because  $g_1(x)$  is the positive root of (54) we have that (55) holds if and only if

$$\min_{x \in I_-^\lambda} \left\{ \frac{K^2}{x^2} + \frac{K}{|x|} \left( \frac{\gamma}{\sqrt{\lambda}} + \sqrt{\lambda} [F(\delta^*) - F(\delta^* + x/\sqrt{\lambda})] \right) - \gamma s F(\delta^* + x/\sqrt{\lambda}) \right\} \geq 0,$$

and a sufficient condition for this to be the case is that

$$\min_{x \in I_-^\lambda} \left\{ \frac{K}{|x|} \sqrt{\lambda} [F(\delta^*) - F(\delta^* + x/\sqrt{\lambda})] - \gamma s(1-s) \right\} \geq 0. \quad (57)$$

By the mean value theorem, we have that for any  $x \in I_-^\lambda \cup I_+^\lambda$  there exists  $\hat{\delta}(x) \in [0, 1]$  such that

$$F(\delta^*) - F(\delta^* + x/\sqrt{\lambda}) = -\frac{x F'(\hat{\delta}(x))}{\sqrt{\lambda}}, \quad (58)$$

and substituting this expression into (57) shows that a sufficient condition for the validity of equation (55) is that we have

$$K \geq K^* \equiv \max_{\delta \in [0,1]} \frac{\gamma s(1-s)}{F'(\delta)}.$$

Because the derivative of the distribution of utility types is assumed to be bounded away from zero on the whole interval  $[0, 1]$ , we have that  $K^*$  is finite and equation (55) follows. One obtains the same constant when applying the same calculations to the function  $g_0(x)$  over the interval  $I_+^\lambda$ .

Now let us turn to the second part of the statement and fix an arbitrary  $\bar{x} \in I_+^\lambda \cap (-I_-^\lambda)$ . Because the function  $g_1(x)$  is the positive root of (54) we have that (56) holds if and only if

$$\max_{x \in I_+^\lambda \cap [\bar{x}, \infty)} \left\{ k^2 x^2 + kx \left( \frac{\gamma}{\sqrt{\lambda}} + \sqrt{\lambda} [F(\delta^*) - F(\delta^* + x/\sqrt{\lambda})] \right) - \gamma s F(\delta^* + x/\sqrt{\lambda}) \right\} \leq 0.$$

Combining this inequality with equation (58) then shows that a sufficient condition for the validity of equation (56) is given by

$$k \leq k^* \equiv \inf_{\delta \in [0,1]} \left( F'(\delta) - \frac{\gamma}{\bar{x}\sqrt{\lambda}} \right),$$

and the desired result now follows by noting that, because the derivative of the distribution of utility types is assumed to be strictly positive on the whole interval  $[0, 1]$ , we can pick the meeting rate  $\lambda$  large enough for the constant  $k^*$  to be strictly positive. One obtains the same constant when applying the same calculations to the function  $g_0(x)$  over the interval  $I_-^\lambda \cap (-\infty, -\bar{x}]$ . ■

**Lemma A.12** *Assume that the conditions of Proposition 7 hold. Then*

$$\lim_{\lambda \rightarrow \infty} D(\lambda) = \int_{-\infty}^0 \frac{\theta_0 g(x) dx}{\theta_0 g(x) + \theta_1 g(-x)} \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} P(\lambda) = \int_0^{\infty} \frac{\theta_1 g(-x) dx}{\theta_0 g(x) + \theta_1 g(-x)},$$

where the function  $g(x)$  is defined as in Lemma A.10.

**Proof.** By Lemma A.10 we have that the integrand

$$H(x; \lambda) \equiv \mathbf{1}_{\{x \in I_{\pm}^{\lambda}\}} \left( \frac{\gamma F(\delta^* + x/\sqrt{\lambda}) + \theta_0 \sqrt{\lambda} g_1(x)}{r + \gamma + \theta_0 \sqrt{\lambda} g_1(x) + \theta_1 \sqrt{\lambda} g_0(x)} \right)$$

in the definition of  $D(\lambda)$  satisfies

$$\lim_{\lambda \rightarrow \infty} H(x; \lambda) = \frac{\theta_0 g(x)}{\theta_0 g(x) + \theta_1 g(-x)}. \quad (59)$$

Now fix an arbitrary  $\bar{x} \in I_+^{\lambda} \cap (-I_-^{\lambda})$  and let the meeting rate  $\lambda$  be large enough. On the interval  $[-\bar{x}, 0]$ , we can bound the integrand above by 1 and below by zero, while on the interval  $I_-^{\lambda} \setminus [-\bar{x}, 0]$  we can use the bounds provided by Lemma A.11 to show that

$$0 \leq H(x; \lambda) \leq \frac{\gamma|x| + \theta_0 \sqrt{\lambda} K}{\sqrt{\lambda}(\theta_0 K + \theta_1 k|x|^2)} \leq \frac{\gamma \delta^* + \theta_0 K}{\theta_0 K + \theta_1 k|x|^2},$$

where the inequality follows from the definition of  $I_{\pm}^{\lambda}$ . Combining these bounds shows that the integrand is bounded by a function that is integrable on  $\mathbb{R}_-$  and does not depend on  $\lambda$ . This allows us to apply the dominated convergence theorem, and the result for  $D(\lambda)$  now follows from (59). The result for the other integral follows from identical calculations. We omit the details. ■

**Proof of Proposition 8.** Assume that there are  $I \geq 2$  utility types  $\delta_1 < \delta_2 < \dots < \delta_I$ , identify the marginal type with the index  $m \in \{1, \dots, I\}$  such that:

$$1 - F(\delta_m) \leq s < 1 - F(\delta_{m-1})$$

and set  $\delta_0 \equiv 0$  and  $\delta_{I+1} \equiv 1$ . Assume further that  $1 - F(\delta_m) < s$ , which occurs generically when the distribution of utility types is restricted to be discrete. Under these assumptions, the same algebraic manipulations that we used to establish (49) and (50) show that we have

$$\Phi_1(\delta) = \Phi_1(\delta_i) = \begin{cases} \frac{1}{\lambda} \frac{\gamma s F(\delta_i)}{1-s-F(\delta_i)} + o\left(\frac{1}{\lambda}\right) & \text{if } i < m \\ F(\delta_i) - (1-s) + \frac{1}{\lambda} \frac{\gamma(1-F(\delta_i))(1-s)}{F(\delta_i)-(1-s)} + o\left(\frac{1}{\lambda}\right) & \text{if } i \geq m, \end{cases} \quad (60)$$

for all  $\delta \in [\delta_i, \delta_{i+1})$  and  $i \in \{1, \dots, I\}$ . Likewise, we have that the local surplus satisfies

$$\sigma(\delta) = \sigma(\delta_i) = \begin{cases} \frac{1}{\lambda \theta_1 (1-s-F(\delta_i))} + o\left(\frac{1}{\lambda}\right) & \text{if } i < m \\ \frac{1}{\lambda \theta_0 (F(\delta_i)-(1-s))} + o\left(\frac{1}{\lambda}\right) & \text{if } i \geq m \end{cases}$$

for all  $\delta \in [\delta_i, \delta_{i+1})$  and  $i \in \{1, \dots, I\}$ , and it follows that the steady-state reservation values satisfy

$$\Delta V(\delta_m) - \Delta V(\delta_i) = \sum_{j=i}^{m-1} (\delta_{j+1} - \delta_j) \sigma(\delta_j) = \sum_{j=i}^{m-1} \frac{\delta_{j+1} - \delta_j}{\theta_1 (1-s-F(\delta_j))} + o\left(\frac{1}{\lambda}\right).$$

for every  $i < m$ , and

$$\Delta V(\delta_i) - \Delta V(\delta_m) = \sum_{j=m}^{i-1} (\delta_{j+1} - \delta_j) \sigma(\delta_j) = \sum_{j=m}^{i-1} \frac{\delta_{j+1} - \delta_j}{\theta_0 (F(\delta_j) - (1-s))} + o\left(\frac{1}{\lambda}\right).$$

for every  $i > m$ . To complete the proof we calculate the steady-state reservation value  $\Delta V(\delta_m)$  of the marginal investor using formula (13). This gives

$$\begin{aligned} r\Delta V(\delta_m) &= \delta_m + \sum_{i=m}^I (\delta_{i+1} - \delta_i) \frac{\gamma(1 - F(\delta_i)) + \lambda\theta_1(1 - s - \Phi_0(\delta_i))}{r + \gamma + \lambda\theta_0\Phi_1(\delta_i) + \lambda\theta_1(1 - s - \Phi_0(\delta_i))} \\ &\quad - \sum_{i=0}^{m-1} (\delta_{i+1} - \delta_i) \frac{\gamma F(\delta_i) + \lambda\theta_0\Phi_1(\delta_i)}{r + \gamma + \lambda\theta_0\Phi_1(\delta_i) + \lambda\theta_1(1 - s - \Phi_0(\delta_i))} \\ &= \delta_m + \frac{1}{\lambda} \sum_{i=m}^I (\delta_{i+1} - \delta_i) \frac{\gamma(1 - F(\delta_i))(F(\delta_i) - (1-s)(1-\theta_1))}{(F(\delta_i) - (1-s))^2} \\ &\quad - \frac{1}{\lambda} \sum_{i=0}^{m-1} (\delta_{i+1} - \delta_i) \frac{\gamma F(\delta_i)(1 - F(\delta_i) - s(1-\theta_0))}{(F(\delta_i) - (1-s))^2} + o\left(\frac{1}{\lambda}\right) \end{aligned}$$

where the second equality follows from condition (1) and the asymptotic expansion of  $\Phi_1(\delta)$  given in equation (60).  $\blacksquare$

**Proof of Proposition 9.** The result follows from Lemmas A.13, A.14, and A.15. To simplify the presentation we assume without loss of generality in these lemmas that the endpoints of the support of the distribution of utility types are given by  $\underline{\delta} = 0$  and  $\bar{\delta} = 1$ .  $\blacksquare$

**Lemma A.13** *Assume that the conditions of Proposition 9 hold true. Then*

$$A(\lambda) \equiv \lambda \int_0^{\delta^*} \sigma(\delta) d\delta - \int_0^{\delta^*} \frac{d\delta}{\frac{r+\gamma}{\lambda} + \theta_1 F'(\delta^*)(\delta^* - \delta) + \Phi_1(\delta)} = O(1) \quad (61)$$

$$B(\lambda) \equiv \lambda \int_{\delta^*}^1 \sigma(\delta) d\delta - \int_{\delta^*}^1 \frac{d\delta}{\frac{r+\gamma}{\lambda} + \theta_0 F'(\delta^*)(\delta^* - \delta) + 1 - s - \Phi_0(\delta)} = O(1). \quad (62)$$

as the meeting rate  $\lambda \rightarrow \infty$ .

**Proof.** To establish (61) we start by noting that

$$\lambda\sigma(\delta) = \frac{\lambda}{r + \gamma + \lambda\theta_1(1 - s - \Phi_0(\delta)) + \lambda\theta_0\Phi_1(\delta)} = \frac{1}{\frac{r+\gamma}{\lambda} + \theta_1(F(\delta^*) - F(\delta)) + \Phi_1(\delta)}, \quad (63)$$

where we used the facts that  $\Phi_0(\delta) = F(\delta) - \Phi_1(\delta)$ , and  $F(\delta^*) = 1 - s$  due to the assumed continuity of the distribution. Substituting this identity into (61), we obtain:

$$|A(\lambda)| \leq \int_0^{\delta^*} \frac{|F'(\delta^*)(\delta^* - \delta) - (F(\delta^*) - F(\delta))|}{\theta_1 F'(\delta^*)(\delta^* - \delta)(F(\delta^*) - F(\delta))} d\delta.$$

Under our assumption that the distribution of utility types is twice continuously differentiable, we can use Taylor's Theorem to extend the integrand by continuity at  $\delta^*$ , with value

$$\lim_{\delta \rightarrow \delta^*} \frac{|F'(\delta^*)(\delta^* - \delta) - (F(\delta^*) - F(\delta))|}{\theta_1 F'(\delta^*)(\delta^* - \delta)(F(\delta^*) - F(\delta))} = \frac{|F''(\delta^*)|}{2\theta_1(F'(\delta^*))^2}$$

Since the derivative is bounded away from zero this shows that the integrand is bounded and (61) follows. Turning to (62) we start by observing that because of (1) and the assumed continuity of the distribution of utility types we have

$$\Phi_1(\delta) = F(\delta) - F(\delta^*) + F(\delta^*) - \Phi_0(\delta) = F(\delta) - F(\delta^*) + 1 - s - \Phi_0(\delta).$$

Substituting this identity into (63) shows that

$$\lambda\sigma(\delta) = \frac{1}{\frac{r+\gamma}{\lambda} + \theta_0(F(\delta) - F(\delta^*)) + 1 - s - \Phi_0(\delta)},$$

and the desired result now follows from the same argument as above. ■

**Lemma A.14** *Assume that the conditions of Proposition 9 hold true. Then*

$$A_0(\lambda) \equiv \int_0^{\delta^*} \frac{d\delta}{\frac{r+\gamma}{\lambda} + \theta_1 F'(\delta^*)(\delta^* - \delta) + \Phi_1(\delta)} = \frac{\log(\lambda)}{2\theta_1 F'(\delta^*)} + O(1).$$

as the meeting rate  $\lambda \rightarrow \infty$ .

**Proof.** To establish a lower bound we start by noting that  $\Phi_1(\delta) \leq \Phi_1(\delta^*)$  for all  $\delta \leq \delta^*$ . Substituting this into the definition of  $A_0(\lambda)$  and integrating we find that

$$\begin{aligned} A_0(\lambda) &\geq \int_0^{\delta^*} \frac{d\delta}{\frac{r+\gamma}{\lambda} + \theta_1 F'(\delta^*)(\delta^* - \delta) + \Phi_1(\delta^*)} \\ &= \left[ \frac{-1}{\theta_1 F'(\delta^*)} \log \left( \frac{r+\gamma}{\lambda} + \theta_1 F'(\delta^*)(\delta^* - \delta) + \Phi_1(\delta^*) \right) \right]_0^{\delta^*} \\ &= \frac{-1}{\theta_1 F'(\delta^*)} \log \left( \sqrt{(\gamma/\lambda)s(1-s)} + o\left(\frac{1}{\sqrt{\lambda}}\right) \right) + O(1) = \frac{\log(\lambda)}{2\theta_1 F'(\delta^*)} + O(1), \end{aligned}$$

where the second equality follows from the asymptotic expansion of  $\Phi_1(\delta^*)$  given in equation (51) above. To establish the reverse inequality let us break down the integral into an integral over the interval  $[0, \delta^* - 1/\sqrt{\lambda}]$ , and an integral over the interval  $[\delta^* - 1/\sqrt{\lambda}, \delta^*]$ . A direct calculation shows that the first integral can be bounded above by:

$$\int_0^{\delta^* - 1/\sqrt{\lambda}} \frac{d\delta}{\theta_1 F'(\delta^*)(\delta^* - \delta)} = \frac{1}{\theta_1 F'(\delta^*)} \log(\delta^* \sqrt{\lambda}) = \frac{\log(\lambda)}{2\theta_1 F'(\delta^*)} + O(1).$$

On the other hand, noting that

$$\inf_{\delta \in [\delta^* - 1/\sqrt{\lambda}, \delta^*]} \Phi_1(\delta) \geq \Phi_1\left(\delta^* - \frac{1}{\sqrt{\lambda}}\right) = \frac{g_1(-1)}{\sqrt{\lambda}}$$

and integrating we find that the second integral can be bounded from above by

$$\int_{\delta^* - 1/\sqrt{\lambda}}^{\delta^*} \frac{d\delta}{\theta_1 F'(\delta^*)(\delta^* - \delta) + g_1(-1)/\sqrt{\lambda}} = \frac{1}{\theta_1 F'(\delta^*)} \log \left( 1 + \frac{\theta_1 F'(\delta^*)}{g_1(-1)} \right) = O(1)$$

where the last equality follows Lemma A.10. ■

**Lemma A.15** *Assume that the conditions of Proposition 9 hold true. Then*

$$B_0(\lambda) \equiv \int_{\delta^*}^1 \frac{d\delta}{\frac{r+\gamma}{\lambda} + \theta_0 F'(\delta^*)(\delta - \delta^*) + 1 - s - \Phi_0(\delta)} = \frac{\log(\lambda)}{2\theta_0 F'(\delta^*)} + O(1)$$

as the meeting rate  $\lambda \rightarrow \infty$ .

**Proof.** The proof is similar to that of Lemma A.14. We omit the details. ■

### Proof of Proposition 10

**Proof.** Integration by part shows that:

$$\int_0^{\delta^*} \Phi_1(\delta) d\delta + \int_{\delta^*}^1 (1 - s - \Phi_0(\delta)) d\delta$$

The quadratic equation for the equilibrium distribution and the assumed continuity of the distribution of utility types jointly imply that

$$\lambda \Phi_1(\delta) = \frac{\gamma s F(\delta)}{\gamma/\lambda + \Phi_1(\delta) + F(\delta^*) - F(\delta)},$$

and combining this identity with arguments similar to those we used in the proof of Lemma A.13 shows that the first integral in the definition of the welfare cost satisfies

$$\left| \int_0^{\delta^*} \left( \lambda \Phi_1(\delta) - \frac{\gamma s F(\delta^*)}{\gamma/\lambda + \Phi_1(\delta) + F'(\delta^*)(\delta^* - \delta)} \right) d\delta \right| = O(1). \quad (64)$$

On the other hand, the same arguments as in the proof of Lemma A.14 imply that

$$\int_{\delta^* - 1/\sqrt{\lambda}}^{\delta^*} \frac{\gamma s F(\delta^*) d\delta}{\gamma/\lambda + \Phi_1(\delta) + F'(\delta^*)(\delta^* - \delta)} \leq \frac{\gamma s F(\delta^*)}{F'(\delta^*)} \log \left( 1 + \frac{F'(\delta^*)}{g_1(-1)} \right) = O(1)$$

and combining this inequality with (64) gives

$$\int_0^{\delta^*} \lambda \Phi_1(\delta) d\delta = \int_0^{\delta^* - 1/\sqrt{\lambda}} \frac{\gamma s F(\delta^*)}{\gamma/\lambda + \Phi_1(\delta) + F'(\delta^*)(\delta^* - \delta)} d\delta + O(1).$$

To obtain a lower bound for the integral, we can bound  $\Phi_1(\delta)$  above by  $\Phi_1(\delta^* - 1/\sqrt{\lambda})$ , and to obtain an upper bound, we can bound  $\Phi_1(\delta)$  below by zero. In both cases, we can compute the resulting integral

explicitly and we find that the upper and the lower bound can both be written as

$$\frac{\gamma s F(\delta^*)}{2F'(\delta^*)} \log(\lambda) + O(1) = \frac{\gamma s(1-s)}{2F'(\delta^*)} \log(\lambda) + O(1).$$

Going through the same steps shows that the second integral satisfies

$$\int_{\delta^*}^1 \lambda(1-s-\Phi_0(\delta)) d\delta = \frac{\gamma s(1-s)}{2F'(\delta^*)} \log(\lambda) + O(1)$$

and the desired result now follows by adding up the asymptotic expansions of the two integrals. In order to complete the proof assume that the distribution of utility types is discrete. Using the same notation as in the proof of Proposition 8 we find that

$$C(\lambda) = \sum_{i=0}^{m-1} (\delta_{i+1} - \delta_i) \Phi_1(\delta_i) + \sum_{i=m}^I (\delta_{i+1} - \delta_i) (1-s-F(\delta_i) + \Phi_1(\delta_i))$$

and the desired conclusion follows from the expansion of  $\Phi_1(\delta_i)$  given in equation (60). ■

## B Non-stationary initial conditions

Assume that the initial distribution of utility types in the population is given by an arbitrary cumulative distribution function  $F_0(\delta)$ , which need not even be absolutely continuous with respect to  $F(\delta)$ . Since the reservation values of Proposition 2 are valid for any joint distribution of types and asset holdings, we need only to determine the evolution of the equilibrium distributions in order to derive the unique equilibrium.

Consider first the distribution of utility types in the whole population. Since upon a preference shock each agent draws a new utility type from  $F(\delta)$  with intensity  $\gamma$ , we have that

$$\dot{F}_t(\delta) = \gamma(F(\delta) - F_t(\delta)).$$

Solving this ordinary differential equation shows that the cumulative distribution of utility types in the whole population is explicitly given by

$$F_t(\delta) = F(\delta) + e^{-\gamma t} (F_0(\delta) - F(\delta))$$

and converges to the long-run distribution  $F(\delta)$  in infinite time. On the other hand, the same arguments as in Section 3.2 show that in equilibrium the distributions of perceived growth rate among the population of asset owners solves the differential equation

$$\dot{\Phi}_{1,t}(\delta) = -\lambda \Phi_{1,t}(\delta)^2 - \lambda(1-s-F_t(m) + \Phi_{1,t}(\delta)) + \gamma(sF(\delta) - \Phi_{1,t}(\delta)).$$

Given an initial condition satisfying the accounting identity

$$\Phi_{0,0}(\delta) + \Phi_{1,0}(\delta) = F_0(\delta)$$

this Riccati equation admits a unique solution that can be expressed in terms of the confluent hypergeometric

function of the first kind  $M_1(a, b; x)$  (see [Abramowitz and Stegun \(1964\)](#)) as

$$\lambda\Phi_{1,t}(\delta) = \lambda(F_t(m) - \Phi_{0,t}(\delta)) = \frac{\dot{Y}_{+,t}(\delta) - A(\delta)\dot{Y}_{-,t}(\delta)}{Y_{+,t}(\delta) - A(\delta)Y_{-,t}(\delta)} \quad (65)$$

with

$$\begin{aligned} Y_{\pm,t}(\delta) &= e^{-\lambda Z_{\pm}(\delta)t} W_{\pm,t}(\delta) \\ Z_{\pm}(\delta) &= \frac{1}{2} (1 - s + \gamma/\lambda - F(\delta)) \pm \frac{1}{2} \Lambda(\delta) \\ W_{\pm,t}(\delta) &= M_1 \left( \frac{\lambda}{\gamma} Z_{\pm}(\delta), 1 \pm \frac{\lambda}{\gamma} \Lambda(m); e^{-\gamma t} \frac{\lambda}{\gamma} (F(\delta) - F_0(\delta)) \right) \end{aligned} \quad (66)$$

and

$$A(\delta) = \frac{\dot{Y}_{+,0}(\delta) - \lambda\Phi_{1,0}(\delta)Y_{+,0}(\delta)}{\dot{Y}_{-,0}(\delta) - \lambda\Phi_{1,0}(\delta)Y_{-,0}(\delta)}.$$

The following lemma relies on standard properties of confluent hypergeometric functions to show that the above cumulative distribution function converges to the same steady-state distribution as in the case with stationary initial condition.

**Lemma B.1** *The equilibrium distributions defined by (65) satisfy  $\lim_{t \rightarrow \infty} \Phi_{q,t}(\delta) = \Phi_q(\delta)$  for any initial distributions  $F_0(\delta)$  and  $F_{1,0}(\delta)$ .*

**Proof.** Straightforward algebra shows that (65) can be rewritten as

$$\lambda\Phi_{1,t}(\delta) = \frac{\lambda Z_+(\delta)W_{+,t}(\delta) - \dot{W}_{-,t}(\delta) + e^{\lambda\Lambda(\delta)t}A(\delta)(\dot{W}_{+,t}(\delta) - \lambda Z_-(\delta)W_{-,t}(\delta))}{e^{\lambda\Lambda(\delta)t}A(\delta)W_{-,t}(\delta) - W_{+,t}(\delta)}.$$

On the other hand, using standard properties of the confluent hypergeometric function of the first kind it can be shown that we have

$$\lim_{\delta \rightarrow \infty} \dot{W}_{\pm,t}(\delta) = \lim_{\delta \rightarrow \infty} (1 - W_{\pm,t}(\delta)) = 0$$

and combining these identities we deduce that

$$\lim_{\delta \rightarrow \infty} \lambda\Phi_{1,t}(\delta) = -\lambda Z_-(\delta) + \lim_{\delta \rightarrow \infty} \frac{\dot{W}_{+,t}(\delta)}{W_{-,t}(\delta)} = -\lambda Z_-(\delta) = \lambda\Phi_1(\delta),$$

where the last equality follows from (66) and the definition of the steady-state distribution  $\Phi_1(\delta)$ . ■

Given the joint distribution of types and asset holdings the equilibrium can be computed by substituting the equilibrium distributions into (12) and (13), and the same arguments as in the stationary case show that this equilibrium converges to the same steady-state equilibrium as in Theorem 1.