Meeting Technologies and Optimal Trading Mechanisms in Competitive Search Markets: Online Appendix

Benjamin Lester*
Federal Reserve Bank of Philadelphia

Ludo Visschers
University of Edinburgh
& Universidad Carlos III, Madrid

Ronald Wolthoff
University of Toronto

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1 Free Entry

Although Proposition 2 establishes that the equilibrium is efficient for any arbitrary ratio of buyers to sellers, a traditional concern in much of the search literature is whether efficiency is also achieved when this ratio is determined endogenously. To address this issue, suppose that sellers can freely enter the market by paying a cost k, as in standard search models (see, e.g., Pissarides, 1985). The planner will then choose the measure of sellers μ_S to maximize net social surplus,

$$\max_{\mu_S \in (0,\infty)} \mu_S \left[S\left(\frac{\mu_B}{\mu_S}\right) - k \right], \tag{1}$$

while the equilibrium measure of sellers follows from the indifference condition

$$R\left(r^*, t^*, \frac{\mu_B}{\mu_S}\right) - k = y. \tag{2}$$

The following lemma states that the market tightness achieved in equilibrium indeed coincides with the solution to the planner's problem.

Lemma A1. *The market equilibrium with free entry is constrained efficient.*

*The views expressed here are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of Philadelphia or the Federal Reserve System.

Proof. The proof resembles Lester et al. (2013) and Albrecht et al. (2014). The equilibrium measure of sellers is determined by the free entry condition (2). Since $S(\lambda) = R(r^*, t^*, \lambda) + \lambda U(r^*, t^*, \lambda) - y$, this condition is equivalent to

$$S\left(\frac{\mu_B}{\mu_S}\right) - k = \frac{\mu_B}{\mu_S} U\left(r^*, t^*, \frac{\mu_B}{\mu_S}\right). \tag{3}$$

The efficient measure of sellers follows from the first-order condition of net social surplus, described in (1), with respect to μ_S. After simplification, this yields

$$S\left(\frac{\mu_B}{\mu_S}\right) - k = \frac{\mu_B}{\mu_S} S'\left(\frac{\mu_B}{\mu_S}\right). \tag{4}$$

Since $S'\left(\frac{\mu_B}{\mu_S}\right) = U\left(r^*, t^*, \frac{\mu_B}{\mu_S}\right)$, it follows that the solution to (3) solves (4). ■

This extends some of the results of Albrecht et al. (2012, 2014) to arbitrary meeting technologies. Clearly, our results regarding the importance of meeting fees and (invariance of) the meeting technology are robust to endogenizing the measure of sellers, as they hold for arbitrary values of the buyer-seller ratio.

2 Necessity of Meeting Fees

Proposition 3 establishes that all equilibria must be payoff-equivalent to the one with second-price auctions and meeting fees. In this proposition, payoff-equivalence concerns the expected payoffs and not the realized payoffs, as it is straightforward to change the latter while keeping the former the same, e.g. by changing the auction format. This observation may raise the question whether one can construct an equilibrium mechanism in which the meeting fees/subsidies are replaced by an extra payment by/to the trading buyer, such that buyers who do not trade receive a zero payoff, even when the meeting technology is not invariant. Lemma A2 establishes that this is not feasible. Before presenting the formal argument, we sketch the intuition.

We know that any mechanism that is chosen in equilibrium has to implement the planner's solution, and it has to be payoff equivalent to the mechanism we describe. This puts a lot of structure on the mechanism: it pins down the probability that a buyer is awarded the good, and it pins down the expected transfer he pays. Suppose we divide the game into three stages: at stage 1, buyers choose a seller; at stage 2, buyers arrive and are potentially asked to pay a meeting fee (or receive a subsidy); and at stage 3, buyers learn their type and report it, whereupon the good is allocated and additional transfers occur. Our mechanism specifies that all buyers pay t at stage 2, and then a second price auction occurs at stage 3. The question can then be phrased: is there an alternative mechanism with *no* transfers at stage 2, and only a transfer between the seller and the buyer who receives the good at stage 3?

Loosely speaking, the reason the answer is "no" is that transfers at stage 3 have to respect both an individual rationality (or participation) constraint and an incentive compatibility (or truth-telling) constraint. If the seller did not charge t at stage 2, he would have to charge higher prices at stage 3; otherwise his revenue would be different, which violates payoff-equivalence. However, he cannot raise prices on only buyers who report high valuations, as this would violate incentive compatibility. Therefore, he would have to raise prices on *all* buyers, but this would violate the participation constraint of buyers with valuations very close to y. Hence, the seller *must* extract rents at stage 2; he *cannot* derive a pricing scheme that extracts the same rents but still respects the buyers' IR and IC constraints.

Lemma A2. *All equilibrium mechanisms must include a transfer, (in expectation) equal to t^*, which is paid by/to each of the buyers who arrive at a seller before they learn their valuations*

Proof. As a first step, consider a mechanism $\{\phi_n(x), \tau_n(x)\}$ that assigns a probability of trade $\phi_n(x)$ in exchange for a transfer $\tau_n(x)$ to an agent who reports being of type x when there are n buyers participating in the mechanism.[1] Once a buyer has learned his type, x, he reports a valuation x' which maximizes

$$\phi_n(x')\,x - \tau_n(x')\,. \tag{5}$$

The incentive compatibility (or truth-telling) constraint then requires

$$\phi_n'(x)\,x - \tau_n'(x) = 0. \tag{6}$$

Hence,

$$\begin{aligned}\tau_n(x) &= \int_y^x \tau_n'(\widetilde{x})\,d\widetilde{x} + C_0 = \int_y^x \widetilde{x}d\phi_n(\widetilde{x}) + C_0\\ &= \phi_n(x)\,x - y\phi_n(y) - \int_y^x \phi_n(\widetilde{x})\,d\widetilde{x} + C_0\\ &= \phi_n(x)\,x - \int_y^x \phi_n(\widetilde{x})\,d\widetilde{x} + C_1,\end{aligned} \tag{7}$$

for constants C_0 and C_1, where the second equality follows from integration by parts. Combining (5) and (7) yields

$$\int_y^x \phi_n(\widetilde{x})\,d\widetilde{x} - C_1. \tag{8}$$

[1] This representation of a mechanism constitutes a slight abuse of notation whereby, e.g., $\phi_n(x_1) = \int \ldots \int \phi(x_1, x_2, \ldots, x_n, n)dF(x_2)\ldots dF(x_n)$, where $\phi(x_1, \ldots, x_n, n)$ is defined in the main text.

Hence, in *any* incentive compatible mechanism, the expected payoff to a buyer with valuation x is completely determined by the probability of trade, $\phi_n(x)$, and a constant C_1.

Next, as a second step, we will show that both the probabilities of trade and the constant C_1 are uniquely determined in any equilibrium of our game. To see this, first note that any equilibrium mechanism must be constrained efficient (Proposition 2 in the text), which implies that the good must be allocated to the agent who values it most. Hence,

$$\phi_n(x) = \begin{cases} F^{n-1}(x) & \text{if } x \geq y \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Substituting (9) into (8) and taking expectations over n and x, we find that the ex ante expected utility of a buyer is

$$U = \sum_{n=1}^{\infty} Q_n(\lambda) \left[\int_{\underline{x}}^{\overline{x}} \int_{y}^{x} F^{n-1}(\widetilde{x})\, d\widetilde{x} dF(x) - C_1 \right]. \tag{10}$$

By Proposition 3, $U = \bar{U}(r^*, t^*, \Lambda)$ in any equilibrium, which implies that C_1 is also uniquely determined. Indeed, in our mechanism, $C_1 = t$, the meeting fee.

Finally, as a last step, consider an equilibrium of our game in which $C_1 = t > 0$. Moreover, suppose we decompose $C_1 = C_1^a + C_1^b$, where C_1^a denotes any fees that are paid before buyers learn their type, and C_1^b are transfers made after reporting x. The individual rationality (or participation) constraint of a buyer with valuation x is given by

$$\int_{y}^{x} \phi_n(\widetilde{x})\, d\widetilde{x} - C_1^b \geq 0. \tag{11}$$

Since this constraint must hold for all $x \in [y, \overline{x}]$, it follows that $C_1^b = 0$ in every equilibrium. ■

3 Restrictions on M

Our assumptions about $P_n(\lambda)$ and $m(z;\lambda)$ impose a number of restrictions on $M(\cdot)$.[2]

(i) m being a probability-generating function requires that M is an analytic function.[3]

(ii) $\frac{1}{n!}\frac{\partial^n}{\partial z^n} m(0;\lambda) \in [0,1]$ for all n and all λ requires that $\frac{(-\lambda)^n}{n!} M^{(n)}(\lambda) \in [0,1]$ for all n and λ;

(iii) $m(1;\lambda) = 1$ for all λ requires that $M(0) = 1$;

[2]To keep notation concise, we write "all λ" instead of "all $\lambda \in (0,\infty)$", "all n instead of "all $n \in \mathbb{N}_0$", and "all z" instead of "all $z \in [0,1)$".

[3]Probability-generating functions are analytic functions (Sachkov, 1997).

(iv) $m_z(1;\lambda) \in [0,\lambda]$ for all λ requires that $M'(0) \in [-1,0]$;

(v) $m_\lambda(z;\lambda) < 0$ for all z and all λ requires that $M'(\lambda) < 0$ for all λ;

(vi) $m_{\lambda\lambda}(z;\lambda) > 0$ for all z and all λ requires that $M''(\lambda) > 0$ for all λ.

Jointly, these restrictions completely characterize the set of feasible $M(\cdot)$. However, a tighter characterization is possible. For example, restriction (ii) can be tightened as follows

(ii′) M satisfies $\frac{(-\lambda)^n}{n!}M^{(n)}(\lambda) \in (0,1)$ for all n and λ.

Proof. We provide a proof by contradiction. Suppose that there exist a $\widehat{n} \in \mathbb{N}_0$ and a $\widehat{\lambda} > 0$ such that $\frac{(-\widehat{\lambda})^{\widehat{n}}}{\widehat{n}!}M^{(\widehat{n})}\left(\widehat{\lambda}\right) = 0$. As $\widehat{\lambda} > 0$, this implies $M^{(\widehat{n})}\left(\widehat{\lambda}\right) = 0$. Because $M^{(\widehat{n})}(\lambda)$ is continuously differentiable but cannot cross zero, it must be the case that its first derivative is zero in $\lambda = \widehat{\lambda}$ as well, i.e. $M^{(\widehat{n}+1)}\left(\widehat{\lambda}\right) = 0$. By induction, it then follows that all higher derivatives must equal zero in this point, $M^{(n)}\left(\widehat{\lambda}\right) = 0$ for all $n \in \{\widehat{n}, \widehat{n}+1, \widehat{n}+2, \ldots\}$.

With all of its derivatives being zero in $\widehat{\lambda}$, the analytic function $M^{(\widehat{n})}(\lambda)$ must be zero in a neighborhood around $\widehat{\lambda}$. Again by induction, we therefore obtain that $M^{(n)}(\lambda) = 0$ for all $n \in \{\widehat{n}, \widehat{n}+1, \widehat{n}+2, \ldots\}$ and all λ. In that case,

$$\sum_{i=0}^{\widehat{n}-1} P_i(\lambda) = \sum_{i=0}^{\widehat{n}-1} \frac{(-\lambda)^i}{i!} M^{(i)}(\lambda) = 1,$$

which is a differential equation with solution $M(\lambda) = 1 + \sum_{i=1}^{\widehat{n}-1} c_i \lambda^i$, for some coefficients c_i. Because $M'(\lambda) < 0$ rules out the possibility that $M(\lambda)$ is a constant, one obtains that $\lim_{\lambda\to\infty} |M(\lambda)| = +\infty$.

This contradicts the requirement that $M(\lambda) \in [0,1]$ for all λ. Hence, it must be true that $\frac{(-\lambda)^n}{n!}M^{(n)}(\lambda) > 0$ for all n and all λ. This immediately implies that $\frac{(-\lambda)^n}{n!}M^{(n)}(\lambda) < 1$, for all n and all λ, which completes the proof. ■

Note that restriction (ii′) implies restriction (v) and (vi), which are therefore redundant. Hence, the set of feasible M can be characterized by restrictions (i), (ii′), (iii), and (iv).

4 Independence

Another well-known property of the urn-ball meeting technology is that $P_n(\lambda)$ does not only represent the distribution of the total number of buyers at each seller, but also the distribution of the number of competitors a buyer faces when arriving at a seller. In other words, a buyer who meets with a seller has no effect on the distribution (and thus the expectation) of the number of other buyers at the same seller. We will say that a meeting technology that satisfies this property exhibits “independence”.

Formally, independence means $Q_n(\lambda) = (1 - Q_0(\lambda)) P_{n-1}(\lambda)$ for all λ and $n \in \mathbb{N}_1$. This property can be shown to be satisfied if and only if the meeting technology is of the following form:[4]

$$P_n(\lambda) = e^{-(1-Q_0(\lambda))\lambda} \frac{[(1 - Q_0(\lambda))\lambda]^n}{n!}. \tag{12}$$

The following lemma establishes that independence is neither a necessary nor a sufficient condition for invariance.

Lemma 1. *Invariance does not imply independence and independence does not imply invariance.*

To see why invariance does not imply independence, consider the geometric technology. As we established in the main text, this meeting technology is invariant. However, this technology is not independent, as the fact that an individual buyer meets with a seller changes the probability distribution over the number of other buyers who meet with the seller. In particular, when a buyer meets with a seller, he learns that the seller has only met with buyers thus far, which increases the expected number of other buyers that this seller will ultimately meet; that is, the conditional mean of the queue length is greater than the unconditional mean.

Finally, to understand why independence does not imply invariance, consider the following variation on the urn-ball technology, in which a longer queue length reduces each buyers’ chances of meeting a seller. That is,[5]

$$P_n(\lambda) = e^{-\Phi(\lambda)\lambda} \frac{[\Phi(\lambda)\lambda]^n}{n!}, \tag{13}$$

where $\Phi(\lambda) : [0, \infty) \to [0, 1]$ satisfies $\Phi(0) = 1$ and $\Phi'(\lambda) < 0$. To understand this meeting technology, imagine that buyers and sellers in each sub-market begin on separate islands. The measure σ of sellers each send a boat to transport the measure β of buyers, so that each boat carries $\lambda = \beta/\sigma$ buyers to the sellers’ island. However, each boat sinks with probability $1 - \Phi(\lambda)$, so that heavier boats are more likely to sink. Then, the buyers that arrive safely at the sellers’ island randomly select a seller, as in the urn-ball specification.

Since the probability of each boat’s safe passage depends on the ratio of buyers to sellers in the sub-market, this technology does not satisfy the requirements of non-rivalry, and hence is not invariant. However, once buyers arrive, the meeting process ensues according to the standard urn-

[4] By the consistency requirement and induction, the condition implies that $P_n(\lambda) = \frac{1}{n!}\lambda^n [1 - Q_0(\lambda)]^n P_0(\lambda)$. Since $P_n(\lambda)$ is a probability distribution, we must have $1 = \sum_0^\infty \frac{1}{n!}\lambda^n [1 - Q_0(\lambda)]^n P_0(\lambda) = e^{\lambda[1-Q_0(\lambda)]} P_0(\lambda)$. Solving the second equation for $P_0(\lambda)$ and substituting the solution into the first equation yields (12). Conversely, equation (12) implies the independence condition immediately.

[5] See Kaas (2010) for a related example of this class of meeting technologies.

ball technology, so that the arrival of an individual buyer has no effect on the distribution of other buyers to arrive. Hence, this meeting process satisfies independence.[6]

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[6]This highlights the difference between non-rivalry and independence. Using our analogy from above, non-rivalry is primarily a requirement on the probability that "a boat arrives safely" (i.e., it must be independent of λ), while independence is primarily a condition on the meeting process that occurs "after boats arrive on shore."