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Trading dynamics in decentralized markets with adverse selection ☆

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Abstract

We study a dynamic, decentralized lemons market with one-time entry and characterize its set of equilibria. Our framework offers a theory of how "frozen" markets suffering from adverse selection recover or "thaw" over time endogenously; given an initial fraction of lemons, our model delivers sharp predictions about the length of time it takes for the market to recover, and how prices and the composition of assets in the market behave over this horizon. We use our framework to analyze a form of government intervention introduced during the recent financial crisis in order to help unfreeze the market for asset-backed securities. We find that, depending on the fraction of lemons in the market, such an intervention can speed up *or slow down* market recovery. More generally, our analysis highlights that the success of an intervention in a lemons market depends on both its size *and duration*.
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1. Introduction

Since the seminal work of Akerlof [1], it is well known that the introduction of low-quality assets, or "lemons," into a market with asymmetrically informed buyers and sellers can disrupt the process of trade; the typical result is that sellers with high-quality assets are unwilling to sell at depressed prices, and thus only low-quality assets are exchanged in equilibrium. Given this result, the problem of adverse selection is often used to explain why the market for high-quality assets can break down or *freeze*. However, perhaps surprisingly, much less is known about how and when the exchange of these assets resumes, or how this market *thaws*.

In this paper, we develop a simple model of trade under adverse selection and use it to study how the severity of the lemons problem (i.e., the initial fraction of lemons in the market) affects the patterns of trade over time. In contrast to much of the existing literature, in which unfreezing a market requires an exogenous event or intervention, we incorporate several natural features of many asset markets that allow this process of recovery to occur endogenously. Thus, given any initial fraction of lemons, our model delivers sharp predictions about the length of time it takes for the market to recover, and how prices and the composition of assets remaining in the market behave over this horizon.

We find that the patterns of trade depend systematically on the initial fraction of lemons. When the lemons problem is mild, trades are executed quickly and at relatively uniform prices. On the other hand, when the lemons problem is more severe, trade can take a substantial amount of time and the terms of trade can vary significantly, both across agents and over time. Since our framework describes explicitly how markets can recover over time on their own, it also provides a natural setting to analyze policies aimed at speeding up market recovery. We provide a specific example related to the recent financial crisis and illustrate how our environment can offer unique (and perhaps counter-intuitive) insights into the efficacy of such policy interventions.

We take as a starting point the classic lemons market of Akerlof [1] and make a few simple modifications. First, in order to study how a frozen market can recover over time, the environment must be *dynamic* and equilibria must be *non-stationary*. Hence, we consider a discrete-time, infinite-horizon model in which a fixed set of buyers and sellers have the opportunity to trade in each period. In addition, we assume that agents permanently exit the market after trading, with no new entrants. As a result, a central aspect of our analysis is how the composition of assets remaining in the market evolves over time, and how this interacts with agents' incentives to trade at a particular point in time. Thus, in our model there is a formal sense in which trade may be sluggish because agents are waiting for market conditions to improve, which seems to be an important feature of many frozen markets that cannot be captured in a static or stationary setting.

Second, we focus on markets in which trade is *decentralized*; in contrast to the competitive paradigm, where agents are bound by the law of one price, we assume that buyers and sellers are matched in pairs and decide bilaterally whether to trade and at what price. This assumption is consistent with the trading structure in many important asset markets, such as the markets for asset-backed securities, corporate bonds, derivatives, real estate, and even certain equities.[3]

[3] By now, the literature on decentralized or "over-the-counter" asset markets has grown quite large; see, e.g., Duffie et al. [12], Vayanos and Weill [40], and Lagos and Rocheteau [32].

There are two reasons why these modifications allow for the eventual exchange of high-quality assets. First, there are two mechanisms that can adjust to facilitate trade: the price and, equally important, the time at which a transaction takes place. Second, agents with different quality assets are allowed to trade at different prices.[4] In the context of this environment, we ask: Are all assets—and in particular high-quality assets—eventually bought and sold? If so, how long does it take, and how does this length of time depend on the initial fraction of lemons? How do prices and the composition of assets in the market evolve over time?

Before outlining our results, it is helpful to describe the model in more detail; we provide a full description of the environment and the definition of equilibria in Section 2. The economy starts at $t = 0$ with an equal measure of buyers and sellers. A fraction $q_0 \in (0, 1)$ of sellers possess a single high-quality asset, and the remainder possess a single low-quality asset. The quality of a seller's asset is private information. In each period, all agents receive a stochastic discount factor shock, and then buyers and sellers in the market are randomly and anonymously matched in pairs. Once matched, buyers make a take-it-or-leave-it offer. If the seller accepts, trade ensues and the pair exits the market; if the seller rejects, both agents remain in the market.

We begin by deriving several important properties of all equilibria in Section 3. First, given any q_0, we show that the fraction of high-quality assets in the market increases over time. Intuitively, a seller with a low-quality asset is more likely to accept any offer than a seller with a high-quality asset. Thus, low-quality sellers exit the market at a quicker rate, and average quality improves over time. Second, we establish that all assets are bought and sold—i.e., the *market clears*—in a finite number of periods. Hence, there is a formal sense in which dynamic, decentralized markets that are frozen because of adverse selection will thaw over time.

Then, in Section 4, we use the properties described above to guide our characterization of equilibria. Unfortunately, a complete analytical characterization of the equilibrium set in this non-stationary environment is very difficult. Given this, we proceed in two steps. First, we impose additional structure on the environment by restricting the set of price offers available to buyers. In particular, we assume buyers can choose to offer one of two prices, which are fixed exogenously: a high price that is accepted by all sellers in equilibrium, or a low price that is only accepted by sufficiently impatient sellers with low-quality assets. Importantly, this restriction preserves the main trade-off that buyers face in each meeting, but simplifies the analysis enough to allow for a full analytical characterization of the equilibrium set. Then, as a second step, we solve numerically for equilibria in the original environment with fully flexible prices and demonstrate that all of the main properties of equilibria that we derive analytically are preserved.

Our equilibrium characterization reveals that the amount of time it takes before the market clears depends crucially on the initial fraction of high-quality assets: the equilibrium characterization involves partitioning the interval $(0, 1)$ based on how many periods of trade, k, it takes before all assets are bought and sold, for a given $q_0 \in (0, 1)$. Fig. 1 depicts a typical (very simple) partition. We highlight two interesting features of this equilibrium characterization.

First, there is a natural monotonicity to the equilibrium set: as q_0 gets smaller, it takes longer for the market to clear. Moreover, we show that the expected amount of time it takes to sell a high-quality asset—which we interpret as a measure of the market's *illiquidity*—also increases as q_0 gets smaller. In this sense, our model provides a theory of endogenous liquidity that varies systematically across states of the world and over time.

[4] See Blouin [3] and Moreno and Wooders [34] for more extensive comparisons between centralized and decentralized exchange in a dynamic setting with adverse selection.

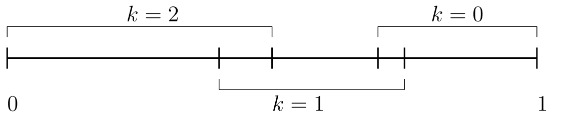

Fig. 1. Number of periods (k) before markets clear for $q_0 \in (0, 1)$.

Second, the equilibrium regions overlap: for some values of q_0, there are multiple equilibria that take different amounts of time for the market to clear. This multiplicity is driven by a complementarity between buyers' actions. When other buyers offer a high price that is accepted by sellers with both high- and low-quality assets, average quality in the ensuing period does not change. Hence, buyers have less incentive to wait for future periods and more incentive to offer a high price now. On the other hand, when other buyers are offering a low price, a larger proportion of sellers with low-quality assets accept this offer and average quality in the future increases. This provides buyers less incentive to offer a high price and trade immediately.

Given our characterization of equilibria, in Section 5 we use our model to study a government program that was introduced in March of 2009 in the hopes of unfreezing the market for asset-backed securities.[5] This program, called the Public–Private Investment Program for Legacy Assets, provided non-recourse loans to buyers willing to purchase these securities, thus reducing the buyers' downside exposure should they discover that they acquired a lemon.

In the context of either static or stationary models of adverse selection, a reduction of downside risk would almost surely ease the lemons problem and help restore liquidity, as this provides buyers with the incentive to offer higher prices. Within the context of our model, we show that such a policy has ambiguous effects on market recovery. Intuitively, an increase in the incentive of buyers to offer higher prices increases both current and future payoffs for sellers holding low-quality assets. If the increase in future payoffs is large relative to the increase in current payoffs, then the owners of low-quality assets have an incentive to delay trade. This slows down the market's recovery, making high-quality assets less liquid and reducing welfare. Thus, our model highlights that the timing—and in particular the *duration*—of any intervention can be just as important as its size. This is a point that has been largely ignored by the literature, in part because most existing work abstracts from non-stationary dynamics (or restricts attention to one-time interventions).

Finally, in Section 6 we discuss the assumption that agents permanently exit the market after trading and how our results might change if we allow for re-entry. Section 7 concludes.

Related literature Our work builds on the literature studying dynamic, decentralized markets with asymmetric information and interdependent values. The majority of this literature restricts attention to stationary equilibria; see, e.g., Inderst [27] and Moreno and Wooders [34]. A notable exception is Blouin [3], who analyzes non-stationary equilibria.[6] In all of these papers, the primary focus is determining what happens to equilibria as frictions vanish. In contrast, our goal is to characterize the set of (non-stationary) equilibria and use this characterization to study how the severity of the lemons problem affects the patterns of trade over time.

[5] Our model captures many of the essential features of this market: trade is decentralized, the fall of housing prices implied substantial heterogeneity in the value of these assets, and in many cases sellers had more information about these assets than potential buyers. We argue each of these points in greater detail in Section 6.

[6] Also see Moreno and Wooders [35].

There is also a large literature that studies the lemons problem in a dynamic setting in which trade is conducted in centralized, competitive markets. Most similar to our paper is Janssen and Roy [28], who also focus on non-stationary equilibria and the patterns of trade over time.[7] In both environments, markets clear in a finite number of periods, and prices and average quality increase over time. In our environment, however, we provide a full characterization of the equilibrium set, analyze the relationship between the initial fraction of high-quality assets and the time it takes for markets to clear, and study how government interventions can affect this relationship.[8]

Our work is also related to the literature on sequential bargaining with asymmetric information and interdependent values.[9] As in our framework, a feature of these models is that buyers use time to screen different types of sellers.[10] However, these models typically have a unique equilibrium, whereas we find multiple equilibria. In Section 4 we discuss how the multiplicity in our environment is driven by the fact that, unlike these models of sequential bargaining, we have a market setting in which all agents are forward-looking.

Lastly, this paper adds to the literature that studies the effect of asymmetric information on market liquidity, along with the more recent literature studying the optimal form of intervention to restore liquidity in frozen markets. Examples of the former literature include the seminal papers by Glosten and Milgrom [20] and Kyle [31], along with more recent contributions by Eisfeldt [15], Kurlat [30], Lester et al. [33], and Rocheteau [37], to name just a few. In terms of the latter literature, our work is perhaps most similar to Chiu and Koeppl [8], who introduce asymmetric information into the random-matching framework of Duffie et al. [12] and characterize steady-state equilibria in which the lemons problem is severe enough to shut down trade.[11] They, too, analyze the effect of policy intervention on trading dynamics, and show how a government purchase of low-quality assets can help to rejuvenate trading activity.

2. The environment

Time is discrete and begins at $t = 0$. There is an equal mass of infinitely lived buyers and sellers. At $t = 0$, each seller possesses a single, indivisible asset, which is either of high (H) or low (L) quality. The fraction of sellers with a high-quality asset at $t = 0$ is $q_0 \in (0, 1)$.

Preferences In every period, each agent's discount factor δ is drawn from a continuously differentiable c.d.f. F with support $[\underline{\delta}, \bar{\delta}]$, where $0 \leq \underline{\delta} < \bar{\delta} < 1$. These draws are i.i.d. across both

[7] Within the context of a stationary environment, many papers study how introducing additional institutions, technologies, or contracts can ease the lemons problem; see, e.g., Hendel et al. [23] and the references therein.

[8] Another important difference between the two papers is that the buyers in Janssen and Roy [28] are somewhat passive: due to free entry of buyers, the equilibrium market price at each date is the expected value of the asset, and thus buyers receive the same payoff in any period. In our model, there are a fixed number of buyers and they face a non-trivial trade-off between current and future payoffs. Not only is this trade-off crucial for the characterization of buyers' optimal behavior, but it also turns out to be an important source of multiplicity in our model. We offer a more detailed discussion of which ingredients in our model help to generate multiple equilibria in Section 4.

[9] See Vincent [41], Evans [16], and Deneckere and Liang [11] for models in which a single seller and a single buyer bargain over time, and Swinkels [38], Hörner and Vieille [25], and Daley and Green [10] for models in which a single long-lived seller faces a sequence of short-lived buyers.

[10] This basic idea goes back to, at least, Wilson [42].

[11] Other recent papers studying the effects of asymmetric information on asset market liquidity and policy interventions include Guerrieri and Shimer [22], Chari et al. [7], Tirole [39], Philippon and Skreta [36], Camargo et al. [6], House and Masatlioglu [26], and Fuchs and Skrzypacz [17].

agents and time. The assumption of random discount factors captures the idea that buyers and sellers have different needs at different times. At a given time, some sellers may need to sell their asset more urgently than others, and some buyers may desire immediate consumption more than others. Across time, each individual agent may be more or less patient in any given period.[12] From a technical point of view, having random discount factors is convenient in that it allows us to focus on pure strategy equilibria; when buyers share the same discount factor, the equilibrium characterization typically involves mixed strategies (see, e.g., Blouin [3]).

When an asset of quality $j \in \{L, H\}$ is transferred from a seller to a buyer, the buyer obtains utility u_j while the seller suffers disutility c_j. This disutility cost can be interpreted as either a direct production cost or as an opportunity cost.[13] For simplicity, we normalize c_L to 0.

Maintained assumptions We assume that

$$u_H > c_H > u_L > c_L = 0. \tag{1}$$

Thus, there are gains from trade in every match, but the price that buyers are willing to pay for a low-quality asset would not be accepted by a high-quality seller, which is necessary to generate the lemons problem. Also, we assume that

$$u_H - c_H > u_L, \tag{2}$$

so that gains from trade in a match with a high-quality seller are greater than in a match with a low-quality seller, which seems to be the natural assumption. Finally, we assume that

$$\underline{\delta} c_H < u_L. \tag{3}$$

We discuss the importance of this assumption below, after we present a few important results. However, at this point, it is worth noting that (3) is a fairly weak assumption; since we impose no additional structure on the shape of the distribution F, the probability that an agent draws a discount factor smaller than u_L/c_H can be made arbitrarily small.

Matching and trade In every period, after agents draw their discount factors, buyers and sellers are randomly and anonymously matched in pairs. Discount factors and the quality of the seller's asset are private information. Once matched, the buyer makes a take-it-or-leave-it offer, which the seller can accept or reject. If a seller accepts, trade ensues and the pair exits the market; there is no entry by additional buyers and sellers. If a seller rejects, no trade occurs and the pair remains in the market. This ensures that there is always an equal measure of buyers and sellers in the market.

[12] Note that all types of agents draw their discount factors from the same c.d.f. F. Though non-essential, we think this is reasonable. For a deeper look at the use of random discount factors, see Higashi et al. [24].

[13] In Camargo and Lester [5], we assume that an asset yields flow payoffs to the seller in every period until it is sold, so that we derive c_j as an opportunity cost explicitly; this alternative specification has no effect on any of the substantive results below. More generally, the reader will note that, as in Duffie et al. [12], buyers and sellers receive different levels of utility from holding a particular asset. This can arise for a multitude of reasons: for example, agents can have different levels of risk aversion, financing costs, regulatory requirements, or hedging needs. In addition, the correlation of endowments with asset returns may differ across agents. The current formulation is a reduced-form representation of such differences. For more discussion and examples in which these differences arise endogenously, see, e.g., Duffie et al. [13], Vayanos and Weill [40], and Gârleanu [18].

Strategies and equilibrium A history for a buyer is the set of all of his past discount factors and (rejected) price offers. However, a buyer has no reason to condition behavior on his history: this history is private information, discount factors are i.i.d., and the probability that he meets his current trading partner in the future is zero, as there is a continuum of agents. Moreover, since there is no aggregate uncertainty, the buyer's history of past offers is not helpful in learning any information about the aggregate state. Thus, a pure strategy for a buyer is a sequence $\mathbf{p} = \{p_t\}_{t=0}^{\infty}$, with $p_t : [\underline{\delta}, \bar{\delta}] \to [0, u_H]$, such that $p_t(\delta)$ is the buyer's offer in period t conditional on still being in the market and drawing discount factor δ. The restriction that $p \le u_H$ is without loss of generality; a buyer would never offer $p > u_H$ in equilibrium.

A history for a seller is the set of all of his past discount factors and all price offers that he has rejected. However, like buyers, a seller has no reason to condition behavior on this history. Thus, a pure strategy for a type j seller is a sequence $\mathbf{a}_j = \{a_t^j\}_{t=0}^{\infty}$, with $a_t^j : [\underline{\delta}, \bar{\delta}] \times [0, u_H] \to \{0, 1\}$, such that $a_t^j(\delta, p)$ is the seller's acceptance decision in period t conditional on still being in the market, drawing discount factor δ, and receiving offer p. We let $a_t^j(\delta, p) = 0$ denote the seller's decision to reject and $a_t^j(\delta, p) = 1$ denote the seller's decision to accept.

We consider symmetric pure-strategy equilibria, which are described by a list $\sigma = (\mathbf{p}, \mathbf{a}_L, \mathbf{a}_H)$. In order to define equilibria, we must determine payoffs at each date t under any strategy profile σ. Though this is a standard calculation when there is a positive measure of agents remaining in the market, we must also specify what happens when there is a zero measure of agents remaining on each side of the market. More specifically, when all remaining agents trade and exit the market in the current period, we must specify the (expected) payoff to an individual should he choose a strategy that results in *not* trading.

In order to avoid imposing ad hoc assumptions, we adopt the following procedure for computing these payoffs. Consider the slightly more general version of our model in which, in each period t, a fraction $\alpha \in (0, 1]$ of the buyers and sellers in the market are matched in pairs, and the remainder do not get the opportunity to trade. The definition of strategies when $\alpha \in (0, 1)$ is the same as when $\alpha = 1$.[14] However, when $\alpha \in (0, 1)$, there is a strictly positive mass of agents remaining in the market for all t, and thus payoffs are *always* well-defined. We define payoffs when $\alpha = 1$ as the limit as α converges to 1 of payoffs when $\alpha < 1$.

More precisely, given a strategy profile σ for all other agents, let $V_t^j(\mathbf{a}|\sigma, \alpha)$ be the expected lifetime payoff to a type j seller in the market in period t following the strategy $\mathbf{a}$ and $V_t^B(\mathbf{p}|\sigma, \alpha)$ be the same payoff to a buyer in the market in period t following the strategy $\mathbf{p}$ when the probability of trade in each period is $\alpha \in (0, 1)$. Both payoffs are computed before discount factors are determined in period t. The payoff to a type j seller in the market in period t following the strategy $\mathbf{a}$ is then $V_t^j(\mathbf{a}|\sigma) = \lim_{\alpha \to 1} V_t^j(\mathbf{a}|\sigma, \alpha)$, while the payoff to a buyer in the market in period t following the strategy $\mathbf{p}$ is $V_t^B(\mathbf{p}|\sigma) = \lim_{\alpha \to 1} V_t^B(\mathbf{p}|\sigma, \alpha)$.

For any strategy profile $\sigma = (\mathbf{p}, \mathbf{a}_L, \mathbf{a}_H)$, denote the probability that a seller of type j in the market in period t accepts an offer p by

$$A_t^j(p|\sigma) = \int a_t^j(\delta, p)\, dF(\delta).$$

Also, denote by $T(\sigma)$ the period in which the market "clears," i.e., the period in which all sellers remaining in the market accept the offers they receive, and set $T(\sigma) = \infty$ if the market

[14] Now a history for a player also includes the periods in which he was able to trade. However, for the same reasons given above, a player has no incentive to condition his behavior on this information.

never clears. Moreover, let $q_t(\sigma)$ denote the fraction of type H sellers in the market in period $t \le T(\sigma^*)$. The sequence $\{q_t(\sigma)\}_{t=0}^{T(\sigma)}$ satisfies the following law of motion:

$$q_{t+1}(\sigma) = \frac{q_t(\sigma)\{1 - \int A_t^H(p_t(\delta)|\sigma)dF(\delta)\}}{q_t(\sigma)\{1 - \int A_t^H(p_t(\delta)|\sigma)dF(\delta)\} + [1 - q_t(\sigma)]\{1 - \int A_t^L(p_t(\delta)|\sigma)dF(\delta)\}}, \tag{4}$$

with $q_0(\sigma) = q_0$. Finally, let $V_t^B(\sigma)$ and $V_t^j(\sigma)$ denote the payoffs to buyers and type j sellers, respectively, when they choose the strategy specified by σ. By definition, if $\sigma = (\mathbf{p}, \mathbf{a}_L, \mathbf{a}_H)$, then $V_t^B(\sigma) = V_t^B(\mathbf{p}|\sigma)$ and $V_t^j(\sigma) = V_t^j(\mathbf{a}_j|\sigma)$.

Definition 1. The strategy profile $\sigma^* = (\{p_t^*\}_{t=0}^\infty, \{a_t^{L*}\}_{t=0}^\infty, \{a_t^{H*}\}_{t=0}^\infty)$, along with the law of motion $\{q_t^*\}_{t=0}^{T(\sigma^*)}$, is an equilibrium if for each $t \le T(\sigma^*)$ and $j \in \{L, H\}$, we have that:

(i) for all $\delta \in [\underline{\delta}, \bar{\delta}]$, $p_t^*(\delta)$ maximizes

$$q_t^*\{A_t^H(p|\sigma^*)[u_H - p] + (1 - A_t^H(p|\sigma^*))\delta V_{t+1}^B(\sigma^*)\} + (1 - q_t^*)\{A_t^L(p|\sigma^*)[u_L - p] + (1 - A_t^L(p|\sigma^*))\delta V_{t+1}^B(\sigma^*)\};$$

(ii) for each $p \in [0, u_H]$ and $\delta \in [\underline{\delta}, \bar{\delta}]$, $a_t^{j*}(\delta, p) = 1$ if, and only if, $p - c_j \ge \delta V_{t+1}^j(\sigma^*)$;
(iii) $q_t^* = q_t(\sigma^*)$.

In words, a strategy profile σ^*, together with a law of motion $\{q_t^*\}_{t=0}^{T(\sigma^*)}$, is an equilibrium if the behavior of each buyer and seller is optimal in every period $t \le T(\sigma^*)$ given $\{q_t^*\}$, and $\{q_t^*\}$ is consistent with the aggregate behavior of all buyers and sellers. The term in (i) is the expected payoff to a buyer in the market in period t when his discount factor is δ and he offers p: when matched with a type j seller, an offer p is accepted with probability $A_t^j(p|\sigma^*)$, yielding the buyer a payoff $u_j - p$, and rejected with probability $1 - A_t^j(p|\sigma^*)$, in which case the buyer's payoff is $\delta V_{t+1}^B(\sigma^*)$. Likewise, the optimal behavior for a seller in the market in period t is to accept an offer of p if, and only if, this offer is at least as high as the payoff from continuing to the following period. For ease of exposition, we say that σ^* is an equilibrium if the pair $(\sigma^*, \{q_t^*\}_{t=0}^{T(\sigma^*)})$ is an equilibrium, where $\{q_t^*\}_{t=0}^{T(\sigma^*)}$ is the law of motion implied by σ^*.

3. Basic properties of equilibria

In this section, we describe the optimal behavior of individual buyers and sellers in our model economy, and examine the aggregate implications of this behavior for equilibrium dynamics.

3.1. Individual behavior

The first fact that we establish is that, in equilibrium, all sellers accept any offer $p \ge c_H$.

Lemma 1. *In any equilibrium σ^*, $A_t^L(p|\sigma^*) = A_t^H(p|\sigma^*) = 1$ for all $t \ge 0$ and $p \ge c_H$.*

Alternatively, any offer $p < c_H$ is always rejected by a type H seller, but it might be accepted by a sufficiently impatient type L seller: in any equilibrium σ^*, a type L seller with discount factor δ accepts an offer $p < c_H$ if, and only if, $\delta < p / V_t^L(\sigma^*)$. The next result follows immediately.

Lemma 2. *In any equilibrium σ^*, $A_t^L(p|\sigma^*) \geq A_t^H(p|\sigma^*) = 0$ for all $t \geq 0$ and $p < c_H$.*

It should be clear from Lemma 1 that a buyer has no incentive to offer $p > c_H$. Hence, the highest continuation payoff possible for a type L seller is c_H. Condition (3) is now easier to understand: it implies that, with strictly positive probability (though it can arbitrarily small), a type L seller accepts an offer of u_L regardless of his continuation payoff.

Lemmas 1 and 2 imply that buyers have a choice. On the one hand, a buyer can make a "high offer" $p = c_H$ that is accepted by both types of sellers. Given a fraction q of the assets in the market are of high quality, this offer yields the buyer a payoff

$$q[u_H - c_H] + (1 - q)[u_L - c_H]. \tag{5}$$

On the other hand, a buyer may want to make a "low offer" $p < c_H$ that can only possibly be accepted by a seller with a low-quality asset. The expected payoff to a buyer from making such an offer depends on the probability that he meets a type L seller, the probability that a type L seller accepts the offer, and the continuation value to the buyer should his offer be rejected. If δ is the buyer's discount factor, then the low offer solves the following problem:

$$\max_p (1 - q) F\left(\frac{p}{v_L}\right)[u_L - p] + \left\{q + (1 - q)\left[1 - F\left(\frac{p}{v_L}\right)\right]\right\}\delta v_B, \tag{6}$$

where v_B is the continuation payoff to the buyer should he not trade, and v_L is the continuation payoff to a seller of type L should he reject the offer p, which occurs with probability $1 - F(p/v_L)$.

In the next result, we establish that price offers are decreasing in the buyers' discount rate; only the most impatient buyers might make a high offer, those who are more patient make offers that are accepted by some low-quality sellers, and the most patient buyers might even offer a price that is rejected by all sellers.[15]

Lemma 3. *Let σ^* be an equilibrium. In any period $t \leq T(\sigma^*)$, there exists $\hat{\delta}_t \in [\underline{\delta}, \bar{\delta}]$ such that a buyer offers $p = c_H$ if, and only if, $\delta \leq \hat{\delta}_t$. Moreover, for all $\delta > \hat{\delta}_t$, the optimal price p is decreasing in δ.*

3.2. Aggregate implications

Given the optimal behavior of buyers and sellers described above, we now establish several implications for equilibrium dynamics. We start with the following result.

Lemma 4. *Suppose the market has not cleared before period T and that the fraction of type H sellers in the market is positive. In any equilibrium, the market clears in period T if, and only if, all buyers in the market offer c_H.*

[15] For ease of exposition, we proceed under the assumption that there is a unique solution to (6). In Appendix A, we extend Lemma 3 to the case in which (6) need not have a unique solution.

Intuitively, if a positive fraction of buyers do not offer c_H, then it must be that they offer $p < c_H$. Since matching is random, some of them will be matched with type H sellers, who always reject such offers, and thus the market will not clear. Hence, the market only clears in the first period T in which all buyers offer c_H.

Now note that for any equilibrium pricing strategy $\{p_t\}_{t=0}^{\infty}$, Lemmas 1 and 2 imply that

$$\int A_t^L\big(p_t(\delta)|\sigma\big)dF(\delta) \geq \int A_t^H\big(p_t(\delta)|\sigma\big)dF(\delta) \tag{7}$$

for all $t \leq T(\sigma^*)$. Given the law of motion for q_t in (4), the next result follows immediately.

Lemma 5. *In any equilibrium σ^*, $\{q_t(\sigma^*)\}_{t=0}^{T(\sigma^*)}$ is increasing.*

Lemmas 4 and 5 suggest that the average quality of assets in the market improves until, perhaps, it is sufficiently high for all buyers to offer c_H, at which time the market clears. We now show that, in any equilibrium, this process is completed in a finite number of periods.

Proposition 1. *The market clears in finite time in any equilibrium.*

Hence, there is a formal sense in which dynamic, decentralized markets that are frozen because of adverse selection eventually thaw over time. The proof of Proposition 1 is in Appendix A. The intuition is as follows. If the market never clears, then the mass of buyers who offer $p < c_H$ is strictly positive in every period t. Moreover, since q_t is increasing, the sequence $\{q_t\}_{t=0}^{\infty}$ converges to some $q_\infty < 1$.[16] However, if $q_\infty < 1$, then the law of motion (4) implies that the fraction of transactions occurring at $p < c_H$ converges to zero, but this is not consistent with optimal behavior. Loosely speaking, when the probability that type L sellers reject an offer $p < c_H$ is close enough to one, it cannot be optimal for buyers to offer $p < c_H$ unless the value of q_∞ is such that

$$q_\infty u_H + (1 - q_\infty)u_L - c_H \leq 0. \tag{8}$$

However, if (8) holds, then buyers will *never* offer $p = c_H$, in which case it cannot be optimal for sellers to reject offers $p < c_H$ with a probability arbitrarily close to one.[17]

Note that the market need not clear in finite time if assumption (3) is violated. Indeed, if $\underline{\delta} c_H > u_L$, then there exist equilibria in which the average quality increases until $q_t u_H + (1 - q_t)u_L = c_H$ for some $t \geq 0$, and remains constant thereafter. In such an equilibrium, once this critical value of q is reached, buyers mix between offering $p = c_H$ and a price that is rejected by both types of sellers for all $t' \geq t$. This is the type of equilibrium that has often been identified in the existing literature (e.g., Blouin [3] and Moreno and Wooders [35]). However, our results suggest that this type of equilibrium is very fragile, as it is not robust to the possibility that a seller sometimes faces the need to sell his asset urgently, even if the likelihood of such an event is arbitrarily small.

16 Since buyers discount the future, we show in Appendix A that there exists a $q^{**} < 1$ such that all buyers find it optimal to offer $p = c_H$ when $q > q^{**}$. Hence, $q_\infty \leq q^{**} < 1$.

17 We show in the proof of Proposition 1 that $q_t < q_\infty$ for all $t < \infty$ when (8) holds.

4. Equilibrium characterization

Given any initial fraction of high-quality assets, $q_0 \in (0,1)$, Proposition 1 suggests that decentralized markets suffering from adverse selection will not remain frozen indefinitely. Instead, our results highlight the fact that these types of markets have built-in mechanisms that can eventually allow them to overcome information frictions without any external intervention or shocks.

However, many important questions remain. From a normative point of view, one would like to know how long this thawing process takes, and how the gains from trade are realized over time. Of particular relevance is the amount of time it takes for a seller with a high-quality asset to trade, which we interpret as a measure of the market's illiquidity. From a positive point of view, one would also like to know how prices and the composition of assets evolve along the transition to market clearing. Answering these questions, and understanding how these answers depend on initial conditions, requires characterizing the set of equilibria.

Unfortunately, characterizing the equilibrium set in the general environment described above is very difficult. To see why, consider an equilibrium with $T(\sigma^*) \geq 2$. Given q_0, deriving $p_0(\delta)$ and $a_0^L(\delta, p)$ requires knowing q_1, and deriving q_1 requires knowing $p_0(\delta)$ and $a_0^L(\delta, p)$ for all $\delta \in [\underline{\delta}, \bar{\delta}]$, so a fixed point problem emerges. Moreover, both of these derivations also require knowing equilibrium payoffs for buyers and type L sellers at $t = 1$ given q_1, which then requires knowing $p_1(\delta)$, $a_1^L(\delta, p)$ and q_2, and so on. Given this interdependence between individual behavior and aggregate conditions—within each period *and* over time—deriving analytical results in a non-stationary environment becomes extremely challenging.

We overcome the challenge of characterizing the equilibrium set in two steps. First, we impose additional structure on the environment by restricting the set of price offers available to buyers. In particular, motivated by our characterization of buyers' optimal behavior in the previous section, we assume buyers can choose to offer one of two prices, which are fixed exogenously: a high price that is accepted by all sellers in equilibrium, or a low price that is only accepted by sufficiently impatient sellers with low-quality assets. Importantly, this restriction preserves the main trade-off that buyers face in each meeting, but eliminates the dependence of the low offer on future payoffs.

Within this more restrictive environment, we are able to provide a complete characterization of the set of equilibria. This characterization provides a clear description of the main properties of equilibria, along with sharp predictions about the evolution of prices and the composition of assets in the market over time.

Then, following the recursive procedure we use to construct equilibria in the environment with two prices, we solve numerically for equilibria in the original environment with fully flexible prices. We demonstrate that the main properties of equilibria that we derive analytically are preserved in the original environment, which should help to convince the reader that our results are not driven by the restriction to two prices.

4.1. Equilibria with restricted price offers

Suppose a buyer's offer p is restricted to the set $\{p_\ell, p_h\}$, where $p_h = c_H$ and p_ℓ is assumed to lie in the interval $(0, u_L)$.[18] Importantly, all of the basic properties of equilibria described in the previous section remain true in this more restrictive environment.

[18] One could imagine, for example, that buyers possess two indivisible objects they can offer, and that these objects are worth p_h and p_ℓ to sellers.

Note that (1) implies that

$$u_H - p_h > u_L - p_\ell. \tag{9}$$

In order to concentrate on the most relevant cases and keep the exposition as clear as possible, we impose two additional assumptions:

$$\underline{\delta} p_h < p_\ell; \tag{10}$$

$$\bar{\delta}(u_H - p_h) \leq u_L - p_\ell. \tag{11}$$

The first assumption is the analog of (3). It implies that a type L seller accepts an offer of p_ℓ with strictly positive probability. Second, since we restrict buyers to offer either p_ℓ or p_h, we want to focus our attention on the region of the parameter space in which they would never prefer to simply not make an offer at all; (11) is a sufficient condition for this to be true.

Constructing the equilibrium set recursively Given the assumptions above, we can provide a complete characterization of the equilibrium set. The first step consists of characterizing the equilibria in which the market clears in the first period of trade, i.e., all buyers offer p_h in $t = 0$. We refer to such equilibria as "0-step" equilibria; more generally, we refer to equilibria in which the market clears in period k as "k-step" equilibria. Then we construct the set of 1-step equilibria, recognizing that such equilibria must have the following properties: (*i*) some agents offer p_ℓ at $t = 0$; and (*ii*) behavior after the first period of trade is given by a 0-step equilibrium. We then repeat this process for $k = 2$, and so on. Since the market clears in finite time in any equilibrium, this recursive procedure exhausts the equilibrium set. All proofs are relegated to Appendix A.

Zero-step equilibria As in the general case, when the fraction of type H sellers in the market is $q \in (0, 1)$, the payoff to a buyer from offering $p_h = c_H$ is

$$\pi_h^B(q) = q[u_H - p_h] + (1 - q)[u_L - p_h]. \tag{12}$$

Similarly, we denote the payoff to a buyer from offering p_ℓ by

$$\pi_\ell^B(q, \delta, v_L, v_B) = (1 - q)F\left(\frac{p_\ell}{v_L}\right)[u_L - p_\ell] + \left\{q + (1 - q)\left[1 - F\left(\frac{p_\ell}{v_L}\right)\right]\right\}\delta v_B. \tag{13}$$

Notice that π_ℓ^B is strictly increasing in both v_B and δ when $\delta v_B > 0$, and non-increasing in v_L.

Let $v_B^0(q_0)$ and $v_L^0(q_0)$ be the payoffs to buyers and type L sellers in a 0-step equilibrium, respectively.[19] It is easy to see that $v_B^0(q_0) = \pi_h^B(q_0)$ and $v_L^0(q_0) \equiv v_L^0 = p_h$. To construct the set of 0-step equilibria, consider the strategy profile σ^0 in which, in every $t \geq 0$, $p_t(\delta) \equiv p_h$ and type L sellers accept an offer p if, and only if, $\delta \leq p/p_h$. It follows from our refinement for computing payoffs when the mass of agents in the market is zero that $V_t^B(\sigma^0) = v_B^0(q_0)$ and $V_t^L(\sigma^0) = v_L^0$ for all $t \geq 1$. Indeed, under σ^0, when the fraction of buyers and sellers who are matched in each period is $\alpha < 1$, all buyers who get the opportunity to trade exit the market, and so the fraction of type H sellers who remain in the market stays the same. Hence, σ^0 is an equilibrium only if

19 In general, we will adopt the convention that a numerical subscript refers to a particular time period, while a numerical superscript refers to the number of periods it takes for the market to clear in equilibrium. In addition, we will use lower case v to denote equilibrium payoffs.

$v_B^0(q_0) > 0$ and all buyers find it optimal to offer p_h in $t = 0$.[20] Since $\pi_\ell^B(q_0, \delta, v_L^0, v_B^0(q_0))$ is strictly increasing in δ, the latter condition is true if, and only if,

$$\pi_h^B(q_0) \geq \pi_\ell^B\big(q_0, \bar{\delta}, v_L^0, v_B^0(q_0)\big). \tag{14}$$

In the proof of Lemma 6, we show that there exists a unique $\underline{q}^0 \in (0, 1)$ such that (14) is satisfied if, and only if, $q_0 \geq \underline{q}^0$. Moreover, we show that $v_B^0(\underline{q}^0) > 0$, and so $v_B^0(q_0) > 0$ for all $q_0 \geq \underline{q}^0$. Thus, σ^0 is an equilibrium if, and only if, $q_0 \in [\underline{q}^0, 1)$. Finally, we also show that (14) is the loosest possible constraint on q_0 that ensures that a buyer finds it optimal to offer p_h at $t = 0$ when all other buyers in the market offer p_h as well. In other words, no strategy profile $\widetilde{\sigma}^0$ such that all buyers offer p_h in $t = 0$ is an equilibrium when (14) is violated.

Lemma 6. *Let $\underline{q}^0 \in (0, 1)$ denote the unique value of q_0 satisfying (14) with equality. There exists a 0-step equilibrium if, and only if, $q_0 \geq \underline{q}^0$.*

Note that $q_0 u_H + (1 - q_0) u_L \geq p_h = c_H$ for any q_0 in the interval $[\underline{q}^0, 1)$, so that p_h corresponds to a market-clearing price in a competitive equilibrium. Thus, when the lemons problem is relatively small, i.e., when q_0 is sufficiently large, the equilibrium outcome in this dynamic, decentralized market coincides with that of a static, frictionless market: trade occurs instantaneously at a single market-clearing price. We will now show, however, that as the lemons problem worsens, equilibrium outcomes no longer resemble those of a centralized competitive market. Instead, these outcomes appear more consistent with models of decentralized trade with search frictions, in the sense that it takes time for buyers and sellers to trade, and they do so at potentially different prices.

One-step equilibria To continue the equilibrium characterization, let

$$q^+(q, v_L) = \frac{q}{q + (1 - q)[1 - F(p_\ell / v_L)]} \tag{15}$$

for any $q \in (0, 1)$. By construction, $q^+(q, v_L)$ is the fraction of type H sellers in the market in the next period if this fraction is q in the current period, a positive mass of buyers offer p_ℓ, and the continuation payoff to a type L seller in case he rejects a price offer is v_L. Since $v_L \leq p_h$, we have that $F(p_\ell / v_L) \geq F(p_\ell / p_h) > 0$, and so $q^+(q, v_L) > q$ for all $q \in (0, 1)$. Also note that $q^+(q, v_L)$ is strictly increasing in q if $p_\ell / v_L < \bar{\delta}$ and that $q^+(q, v_L) \equiv 1$ if $p_\ell / v_L \geq \bar{\delta}$.

Consider a strategy profile σ^1 such that: (*i*) a positive mass of buyers offer p_ℓ in $t = 0$; (*ii*) the fraction of type H sellers in the market in $t = 1$ is q'; and (*iii*) behavior from $t = 1$ on is that of a 0-step equilibrium with $q_0 = q'$. The following conditions are necessary and sufficient for σ^1 to be an equilibrium:[21]

$$q^+\big(q_0, v_L^0\big) = q'; \tag{16}$$

$$q' \geq \underline{q}^0; \tag{17}$$

$$\pi_h^B(q_0) < \pi_\ell^B\big(q_0, \bar{\delta}, v_L^0, v_B^0(q')\big). \tag{18}$$

[20] Assumption (10) implies that if $v_B^0(q_0) = 0$, then buyers find it optimal to offer p_ℓ at $t = 0$.

[21] It must also be the case that the type j sellers accept an offer of $p = 0$ in $t = 0$ if, and only if, $\delta \leq (p - c_j)/p_h$. This optimal behavior will be implicitly assumed throughout the analysis.

The first condition is simply the law of motion for q_t from $t = 0$ to $t = 1$. Since $v_L^0 = p_h$ is a constant, $q^+(q_0, v_L^0)$ is a continuous and strictly increasing function of q_0 specifying the unique implied value of q_1 in a candidate 1-step equilibrium. The second condition follows from Lemma 6. It ensures that the fraction of type H sellers in $t = 1$ falls in the region of 0-step equilibria. The third condition ensures that a positive mass of buyers find it optimal to offer p_ℓ in $t = 0$, given the strategy profile σ^1. Since $\pi_\ell^B(q_0, \delta, v_L^0, v_B^0(q'))$ is strictly increasing in δ, the incentive of a buyer to offer p_ℓ in $t = 0$ when the market clears in $t = 1$ increases with the buyer's patience. As it turns out, (16) and (17) provide a lower bound on the values of q_0 for which a 1-step equilibrium exists, while (18) provides an upper bound. Lemma 7 formalizes these results.

Lemma 7. *Let $\bar{q}^1$ denote the unique value of q_0 satisfying (18) with equality, and define $\underline{q}^1$ to be such that $q^+(\underline{q}^1, v_L^0) = \underline{q}^0$ if $p_\ell/v_L^0 < \bar{\delta}$ and $\underline{q}^1 = 0$ otherwise. Then $\underline{q}^1 < \underline{q}^0 < \bar{q}^1 < 1$ and there exists a 1-step equilibrium if, and only if, $q_0 \in [\underline{q}^1, \bar{q}^1) \cap (0, 1)$. Moreover, for each $q_0 \in [\underline{q}^1, \bar{q}^1) \cap (0, 1)$, there exists a unique $q' \in [\underline{q}^0, 1]$ such that q' is the value of q_1 in any 1-step equilibrium when the initial fraction of type H sellers is q_0.*

In words, if $q_0 = \bar{q}^1$, then the most patient buyer is exactly indifferent between offering p_ℓ and p_h when a positive mass of other buyers are offering p_ℓ. For any $q_0 > \bar{q}^1$, the payoff to such a buyer from immediately trading at price p_h is greater than the payoff from offering p_ℓ and not trading with positive probability. When $p_\ell/v_L^0 < \bar{\delta}$, $\underline{q}^1 > 0$ is the unique value of q_0 such that, if a positive mass of buyers offer p_ℓ, then the fraction of high-quality sellers in the next period is $\underline{q}^0$, the minimum value required for market clearing. If $p_\ell/v_L^0 \geq \bar{\delta}$, so that even the most patient type L seller would rather accept an offer of p_ℓ today than wait one period for an offer of p_h, then $\underline{q}^1 = 0$.

The fact that $\underline{q}^0 < \bar{q}^1$ implies that 0- and 1-step equilibria coexist when $q_0 \in [\underline{q}^0, \bar{q}^1)$. In this region, if all other buyers are offering p_h, then a buyer's payoffs from trading at $t = 0$ and at $t = 1$ are the same, so that it is always optimal to offer p_h. However, if a positive mass of other buyers are offering p_ℓ, then the market does not clear at $t = 0$ and the payoff to trading at $t = 1$ increases (since $q_1 > q_0$), rendering it optimal for a patient buyer to offer p_ℓ and risk trading only in the next period. We continue this discussion of multiple equilibria at the end of the section.

Let $Q_+^1(q_0) \equiv q^+(q_0, v_L^0)$ denote the value of q_1 corresponding to an initial value q_0 in a 1-step equilibrium. Then the payoff to a buyer in a 1-step equilibrium is

$$v_B^1(q_0) = \int \max\{\pi_h^B(q_0), \pi_\ell^B\big(q_0, \delta, v_L^0, v_B^0\big(Q_+^1(q_0)\big)\big)\} dF(\delta).$$

Now let $\mathbb{I}$ denote the indicator function. The payoff to a type L seller in a 1-step equilibrium is

$$v_L^1(q_0) = \xi^1(q_0) p_h + \big(1 - \xi^1(q_0)\big) \int \max\{p_\ell, \delta v_L^0\} dF(\delta),$$

where

$$\xi^1(q_0) = \int \mathbb{I}\{\pi_h^B(q_0) \geq \pi_\ell^B\big(q_0, \delta, v_L^0, v_B^0\big(Q_+^1(q_0)\big)\big)\} dF(\delta)$$

is the fraction of buyers offering p_h at $t = 0$. Lemma 10 in Appendix A establishes that both v_B^1 and v_L^1 are continuous in q_0 and that $\lim_{q_0 \to \bar{q}^1} \xi^1(q_0) = 1$, so that $v_L^1(\bar{q}^1) = \lim_{q_0 \to \bar{q}^1} v_L^1(q_0) = v_L^0$.

Two-step equilibria We now provide a complete characterization of 2-step equilibria. The process of characterizing k-step equilibria is nearly identical for all $k \geq 2$, so the methodology developed here will allow for a complete characterization of equilibria in the next subsection.

Consider a strategy profile σ^2 such that: (i) a positive mass of buyers offer p_ℓ in $t = 0$; (ii) the fraction of type H sellers in the market in $t = 1$ is q'; and (iii) behavior from $t = 1$ on is that of a 1-step equilibrium with $q_0 = q'$. In order for σ^2 to be an equilibrium, it must satisfy the following three necessary and sufficient conditions:

$$q^+\big(q_0, v_L^1(q')\big) = q'; \tag{19}$$

$$q' \in \big[\underline{q}^1, \bar{q}^1\big) \cap (0, 1); \tag{20}$$

$$\pi_h^B(q_0) < \pi_\ell^B\big(q_0, \bar{\delta}, v_L^1(q'), v_B^1(q')\big). \tag{21}$$

The first condition is the analog of condition (16) for 1-step equilibria; it is the law of motion for q_t from $t = 0$ to $t = 1$, conditional on a 1-step equilibrium beginning at $t = 1$. Unlike (16), the fraction q' in (19) is the solution to a fixed point problem: if type L sellers expect continuation payoffs to be that of a 1-step equilibrium in which the initial fraction of type H sellers is q', then the fraction of type L sellers who accept an offer of p_ℓ in $t = 0$ must be such that this conjecture is correct. This fixed point problem does not appear in (16) since the payoff v_L^0 for type L sellers in a 0-step equilibrium is independent of the initial fraction of high-quality assets. The second condition ensures that there exists a 1-step equilibrium at $t = 1$ given an initial fraction q' of high-quality assets. The final condition ensures that a positive mass of buyers find it optimal to offer p_ℓ at $t = 0$ given equilibrium payoffs at $t = 1$.

Since $v_L^1(q') \leq v_L^1(\bar{q}^1) = v_L^0$ for all $q' \in [\underline{q}^1, \bar{q}^1) \cap (0, 1)$, we have that $p_\ell / v_L^0 \geq \bar{\delta}$ implies that $q^+(q_0, v_L^1(q')) = 1$ for all $q' \in [\underline{q}^1, \bar{q}^1) \cap (0, 1)$. Thus, no 2-step equilibrium exists if $p_L / v_L^0 \geq \bar{\delta}$. Suppose then that $p_\ell / v_L^0 < \bar{\delta}$. We show in the proof of Lemma 8 that (19) and (20) imply (21). Intuitively, the incentive of the most patient buyer to choose p_ℓ in $t = 0$ is even greater than his incentive to choose p_ℓ in $t = 1$, when the fraction of type H sellers in the market is $q' > q_0$. Hence, if the most patient buyer strictly prefers to choose p_ℓ in $t = 1$, which is true by (20), then he also strictly prefers to offer p_ℓ at $t = 0$ and (21) is satisfied. Therefore, (19) and (20) are necessary and sufficient conditions for a 2-step equilibrium.

Let $Q_+^2 : q_0 \mapsto q'$ denote the map defined by (19). In the proof of Lemma 8, we also show that $Q_+^2(q_0)$ is a well-defined function that is both continuous and strictly increasing in q_0. Therefore, for any q_0, there is a unique value of q_1 in any candidate 2-step equilibrium. These properties of $Q_+^2(q_0)$ greatly simplify the characterization of 2-step equilibria: the necessary and sufficient conditions (19) and (20) become $Q_+^2(q_0) \geq \underline{q}^1$ and $Q_+^2(q_0) < \bar{q}^1$. We can now state Lemma 8.

Lemma 8. *Suppose that $\bar{\delta} > p_\ell / v_L^0$. Let $\bar{q}^2$ be the unique solution to $q^+(\bar{q}^2, v_L^1(\bar{q}^1)) = \bar{q}^1$, and define $\underline{q}^2$ to be such that $q^+(\underline{q}^2, v_L^1(\underline{q}^1)) = \underline{q}^1$ if $p_\ell / v_L^1(\underline{q}^1) < \bar{\delta}$ and $\underline{q}^2 = 0$ otherwise. Then $\underline{q}^2 < \underline{q}^1 < \bar{q}^2 < \bar{q}^1$ and there exists a 2-step equilibrium if, and only if, $q_0 \in [\underline{q}^2, \bar{q}^2) \cap (0, 1)$. Moreover, for each $q_0 \in [\underline{q}^2, \bar{q}^2) \cap (0, 1)$, there exists a unique $q' \in [\underline{q}^1, \bar{q}^1)$ such that q' is the value of q_1 in any 2-step equilibrium when the initial fraction of type H sellers is q_0.*

Fig. 2 provides some intuition for the equilibrium characterization so far. After deriving $\underline{q}^0$ and $\bar{q}^1$, we identified $\underline{q}^1$ as the value of q_0 that would "land" exactly on $\underline{q}^0$ at $t = 1$ given the law of motion $Q_+^1(q_0)$. Since this law of motion is continuous and strictly increasing in q_0, we

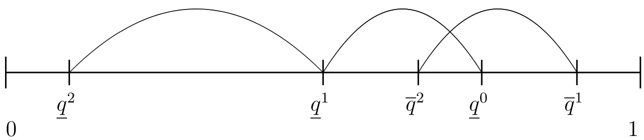

Fig. 2. Deriving bounds on equilibrium regions.

are assured that any $q_0 > \underline{q}^1$ will “land” at $q' > \underline{q}^0$ in a candidate 1-step equilibrium. Moving backwards, we then identified $\underline{q}^2$ and $\bar{q}^2$ as the values of q_0 that would “land” exactly on $\underline{q}^1$ and $\bar{q}^1$, respectively, given the law of motion $Q^2_+(q_0)$. Though this law of motion is slightly more complicated, the fact that it remains continuous and strictly increasing assures us that any $q_0 \in [\underline{q}^2, \bar{q}^2) \cap (0, 1)$ will “land” within the region of 1-step equilibria. Finally, since $v^1_L(\bar{q}^1) = v^0_L$, $\bar{q}^1 > \underline{q}^0$, and

$$q^+\big(\bar{q}^2, v^0_L\big) = \bar{q}^1 > \underline{q}^0 = q^+\big(\underline{q}^1, v^0_L\big),$$

the fact that $\bar{q}^2 > \underline{q}^1$ follows immediately from the fact that $q^+(q_0, v_L)$ is strictly increasing in q_0 for any v_L such that $p_\ell/v_L < \bar{\delta}$. Thus, there is no “gap” in the values of q_0 for which 1-step and 2-step equilibria exist; rather, the multiplicity of equilibria that emerged in the region $[\underline{q}^0, \bar{q}^1)$ also takes place in the region $[\underline{q}^1, \bar{q}^2)$.

To finish the characterization of 2-step equilibria, notice that since $Q^2_+(q_0)$ is uniquely defined, so too are payoffs for type L sellers and buyers in such an equilibrium; Eqs. (23) to (25) below show how to construct the payoffs $v^2_B(q_0)$ and $v^2_L(q_0)$, along with the fraction $\xi^2(q_0)$ of buyers that offer p_h at $t = 0$ in a 2-step equilibrium. Lemma 11 in Appendix A establishes that v^2_B and v^2_L are continuous in q_0 and that $\lim_{q_0 \to \bar{q}^2} \xi^2(q_0) = \xi^1(\bar{q}^2)$, so that $v^2_L(\bar{q}^2) = \lim_{q_0 \to \bar{q}^2} v^2_L(q_0) = v^1_L(\bar{q}^2)$.

A full characterization The characterization of k-step equilibria for $k \geq 3$ proceeds by induction and follows almost exactly the characterization of 2-step equilibria. Hence, for ease of exposition, we just sketch the process here and leave the details for Appendix A.

As the first step, we take as given the range of values of q_0 for which a $(k-1)$-step equilibrium exists, $[\underline{q}^{k-1}, \bar{q}^{k-1}) \cap (0, 1)$, along with the corresponding equilibrium payoffs $v^{k-1}_L(q)$ and $v^{k-1}_B(q)$. Then we define the law of motion $Q^k_+ : q_0 \mapsto q'$ as the solution to the fixed point problem

$$q^+\big(q_0, v^{k-1}_L(q')\big) = q', \tag{22}$$

where $q' \in [\underline{q}^{k-1}, \bar{q}^{k-1}) \cap (0, 1)$. We establish two important results: first, that the most patient buyer strictly prefers to offer p_ℓ at $t = 0$ if (22) is satisfied; and second, that Q^k_+ is continuous and strictly increasing. This implies that a k-step equilibrium exists if, and only if, $q_0 \in [\underline{q}^k, \bar{q}^k) \cap (0, 1)$, where the lower bound $\underline{q}^k$ is such that $Q^k_+(\underline{q}^k) = \underline{q}^{k-1}$ if $p_\ell/v^{k-1}_L(\underline{q}^{k-1}) < \bar{\delta}$ and $\underline{q}^k = 0$ otherwise, and the upper bound $\bar{q}^k$ satisfies $Q^k_+(\bar{q}^k) = \bar{q}^{k-1}$. Moreover, we show that $\underline{q}^k \leq \underline{q}^{k-1} < \bar{q}^k < \bar{q}^{k-1}$, so that there is no gap in the values of q_0 for which k-step and $(k-1)$-step equilibria exist.

We finish the construction of k-step equilibria by determining the payoffs to buyers and type L sellers in such an equilibrium. The payoff to buyers is

$$v_B^k(q_0) = \int \max\{\pi_h^B(q_0), \pi_\ell^B(q_0, \delta, v_L^{k-1}(Q_+^k(q_0)), v_B^{k-1}(Q_+^k(q_0)))\} dF(\delta). \tag{23}$$

The payoff to type L sellers is

$$v_L^k(q_0) = \xi^k(q_0) p_h + (1 - \xi^k(q_0)) \int \max\{p_\ell, \delta v_L^{k-1}(Q_+^k(q_0))\} dF(\delta), \tag{24}$$

where

$$\xi^k(q_0) = \int \mathbb{I}\{\pi_h^B(q_0) \geq \pi_\ell^B(q_0, \delta, v_L^{k-1}(Q_+^k(q_0)), v_B^{k-1}(Q_+^k(q_0)))\} dF(\delta) \tag{25}$$

is the fraction of buyers who offer p_h at $t = 0$. We begin this induction process with $k = 3$, and continue it as long as $p_\ell / v_L^{k-1}(\bar{q}^{k-1}) < \bar{\delta}$, which ensures $\bar{q}^k > 0$ and thus the existence of k-step equilibria. Proposition 2 provides a full characterization of the equilibrium set.

Proposition 2. *There exists $1 \leq K < \infty$ and sequences $\{\underline{q}^k\}_{k=0}^K$ and $\{\bar{q}^k\}_{k=0}^K$, with $\bar{q}^0 = 1$, $\underline{q}^K = 0$, and $\underline{q}^k \leq \underline{q}^{k-1} < \bar{q}^k < \bar{q}^{k-1}$ for all $k \in \{1, \ldots, K\}$, such that a k-step equilibrium exists if, and only if, $q_0 \in [\underline{q}^k, \bar{q}^k) \cap (0, 1)$. Moreover, for each $q_0 \in [\underline{q}^k, \bar{q}^k) \cap (0, 1)$, there exists a unique $q' \in [\underline{q}^{k-1}, \bar{q}^{k-1})$ such that $q' = Q_+^k(q_0)$ is the value of q_1 in any k-step equilibrium when the initial fraction of type H sellers is q_0.*

The payoffs for buyers and type L sellers are uniquely defined in every equilibrium and are determined recursively as follows: (i) $v_B^0(q_0) = \pi_h^B(q_0)$ and $v_L^0(q_0) \equiv p_h$; (ii) for each $k \in \{1, \ldots, K\}$, v_B^k and v_L^k are given by (23) and (24), respectively.

The cutoffs $\{\underline{q}^k\}_{k=0}^{K-1}$ and $\{\bar{q}^k\}_{k=1}^K$ are defined recursively as follows: (i) $\underline{q}^0$ is the unique value of q_0 for which $\pi_h^B(q_0) = \pi_\ell^B(q_0, \bar{\delta}, v_L^0, v_B^0(q_0))$ and, for each $k \in \{1, \ldots, K\}$, $\underline{q}^k$ is such that $q^+(\underline{q}^k, v_L^{k-1}(\underline{q}^{k-1})) = \underline{q}^{k-1}$ if $p_\ell / v_L^{k-1}(\underline{q}^{k-1}) < \bar{\delta}$, and $\underline{q}^k = 0$ otherwise; (ii) $\bar{q}^1$ is the only value of q_0 for which $\pi_h^B(q_0) = \pi_\ell^B(q_0, \bar{\delta}, v_L^0, v_B^0(q^+(q_0, v_L^0)))$ and, for each $k \in \{2, \ldots, K\}$, $\bar{q}^k$ is such that $q^+(\bar{q}^k, v_L^{k-1}(\bar{q}^{k-1})) = \bar{q}^{k-1}$. Finally, $K = \max\{k : p_\ell / v_L^{k-1}(\bar{q}^{k-1}) < \bar{\delta}\}$.

Properties of equilibria Proposition 2 provides a complete characterization of the set of equilibrium. It specifies a sequence of cutoffs that partition the interval $(0, 1)$ into regions such that, for all q_0 in one such region, there exists an equilibrium in which the market takes the same number of periods k to clear. Here, we briefly highlight a few important properties of these equilibria.

First, since $\underline{q}^{k-1} < \bar{q}^k$ for all $k \in \{1, \ldots, K\}$, there is multiplicity of equilibria: for some q_0, there exist equilibria in which markets clear quickly and others where the process takes longer. This multiplicity is driven by a complementarity between buyers' actions. When other buyers offer the high price, average quality in the ensuing period does not change, since sellers with both high- and low-quality assets accept this offer in equal proportion. This gives buyers less incentive to wait for future periods to trade and more incentive to offer a high price now. However, when other buyers offer the low price, a larger proportion of sellers with low-quality assets accept this offer and average quality in the future increases. This provides buyers less incentive to offer a high price and trade immediately.[22] The existence of multiple equilibria for a given q_0 suggests

[22] Since buyers discount the future, the increase in quality between two consecutive periods must be large enough for (at least some) buyers to have an incentive to offer a low price. Hence, the scope for multiplicity of equilibria vanishes

that coordination failures can also contribute to illiquidity in dynamic, decentralized markets with adverse selection.[23]

Second, despite the multiplicity of equilibria, there is still a natural monotonicity to the relationship between the initial fraction of lemons and the amount of time to market clearing: for any $0 < q_0 < q_0' < 1$, if there exists a k-step equilibrium when the initial fraction of high-quality assets is q_0, then there exists a k'-step equilibrium with $k' \leq k$ when the initial fraction of high-quality assets is q_0'. This follows from the fact that $\bar{q}^k$ is strictly decreasing in k, and so an increase in q_0 reduces (weakly) the maximum number of periods it takes for the market to clear. A similar monotonicity emerges when one studies the relationship between the initial fraction of lemons and the expected amount of time it takes to sell a high-quality asset, which we interpret as a measure of *illiquidity*; a liquid market is one where sellers can quickly find a buyer to purchase their high-quality asset (at an acceptable price), whereas an illiquid market is one where this process takes a long time.[24] In Camargo and Lester [5], we establish that for all $k \geq 1$, the expected amount of time it takes to sell a high-quality asset in a k-step equilibrium is decreasing in q_0.[25] Hence, the theory presented here provides a novel theory of liquidity based on adverse selection.

Finally, for any equilibrium, the model provides sharp predictions for the dynamics of prices and allocations. Indeed, consider a k-step equilibrium with initial fraction $q_0 \in [\underline{q}^k, \bar{q}^k) \cap (0, 1)$ of high-quality assets, and define the sequence $\{q_t\}_{t=1}^k$ to be such that $q_t = Q_+^{k-t+1}(q_{t-1})$ for all $t \in \{1, \dots, k\}$. The average price in period $t \in \{0, \dots, k\}$ in a k-step equilibrium is then

$$p_t^{avg} = \xi^{k-t}(q_t) p_h + \left[1 - \xi^{k-t}(q_t)\right] p_\ell.$$

We know from the proof of Proposition 2 that $\xi^s(q) \leq \xi^{s-1}(Q_+^s(q))$ for all $s \in \{0, \dots, k\}$. Hence, p_t^{avg} increases (weakly) over time. In particular, if q_0 is low enough that $\pi_h^B(q_0) < 0$, then $p_t^{avg} = p_\ell$ in early periods, and only low-quality assets exit the market. Eventually the average quality of assets in the market rises enough for some buyers to offer p_h, after which prices continue to increase until all buyers offer p_h and the market clears.

when q_0 is sufficiently close to one. Indeed, as we show in the proof of Proposition 1, there exists $q^{**} < 1$ such that buyers have a strict incentive to offer $p = c_H$ (and thus trade immediately) when $q_0 > q^{**}$.

[23] At the heart of this multiplicity is the fact that the behavior of an individual buyer depends on the future composition of assets in the market, which is determined by the aggregate behavior of buyers. Hence, multiplicity emerges here in part because (*i*) there are many buyers and (*ii*) these buyers are forward-looking. This helps to explain why such multiplicity does *not* arise in related environments in which a seller who has private information bargains with a single buyer (see, e.g., Vincent [41], Evans [16], and Deneckere and Liang [11]) or with a sequence of *myopic* buyers (see, e.g., Hörner and Vieille [25] and the references therein). Note, however, that (*i*) and (*ii*) above are neither necessary nor sufficient for multiplicity (see, e.g., Gerardi et al. [19]).

[24] An asset is typically considered liquid if it can be sold quickly and at little discount. In many models, trade is instantaneous by construction, and thus the only measure of liquidity is the difference between the actual market price and the price in some frictionless benchmark. In these models, time is a margin that simply does not adjust; see, e.g., Amihud and Mendelson [2], Constantinides [9], Kyle [31], Glosten and Milgrom [20], Eisfeldt [15], and Duffie et al. [12]. In the current model, the opposite is true: since all high-quality assets sell at $p_h = c_H$ in equilibrium, the appropriate measure of liquidity for these assets is the expected amount of time it takes to sell them.

[25] Due to the presence of multiple equilibria, comparing liquidity across all values of $q_0 \in (0, 1)$ is more complicated. Nonetheless, a similar monotonicity emerges when one uses certain equilibrium selection rules; for example, one can show a reduction in q_0 increases the *minimum* expected amount of time it takes to sell a high-quality asset.

4.2. Equilibria with unrestricted price offers

We now solve numerically for equilibria in the original environment with unrestricted offers and show that the main qualitative properties of the equilibria with restricted prices offers are preserved.

To start, note that the payoff to a buyer from offering $p_h = c_H$ is still given by $\pi_h^B(q)$, as defined in (12). However, if a buyer prefers to make a low offer, then he chooses an offer $p_\ell(\delta, v_L, v_B)$ that solves (6); let $\tilde{\pi}_\ell^B(q, \delta, v_L, v_B)$ denote the buyer's payoff from offering $p_\ell(\delta, v_L, v_B)$. The same argument used to establish Lemma 6 shows that a 0-step equilibrium exists if, and only if, $q_0 \in [\underline{q}^0, 1)$, where $\underline{q}^0 \in (0, 1)$ is the only solution to

$$\pi_h^B(\underline{q}^0) = \tilde{\pi}_\ell^B(\underline{q}^0, \bar{\delta}, v_L^0, v_B^0(\underline{q}^0)).$$

Now, in order to describe the k-step equilibria for all $k \geq 1$, let

$$q^+(q, v_L, v_B) = \frac{q}{q + (1-q)\{1 - \frac{1}{1-F(\hat{\delta})}\int_{\hat{\delta}}^{\bar{\delta}} F(\frac{p_\ell(\delta, v_L, v_B)}{v_L}) dF(\delta)\}}, \tag{26}$$

where $\hat{\delta} = \hat{\delta}(q, v_L, v_B) = \inf\{\delta \in [\underline{\delta}, \bar{\delta}] : \tilde{\pi}_\ell^B(q, \delta, v_L, v_B) \geq \pi_h^B(q)\}$. The law of motion in (26) is the analog of (15) for the case of unrestricted price offers. The same argument as in the case with restricted price offers shows that, for all $k \geq 1$, if (*i*) $[\underline{q}^{k-1}, \bar{q}^{k-1})$ is the range of values of q_0 for a which a $(k-1)$-step equilibrium exists, and (*ii*) $v_L^{k-1}(q)$ and $v_B^{k-1}(q)$ are the associated equilibrium payoffs, then the following conditions are necessary and sufficient for a k-step equilibrium:

$$q^+(q_0, v_L^{k-1}(q'), v_B^{k-1}(q')) = q'; \tag{27}$$

$$q' \in [\underline{q}^{k-1}, \bar{q}^{k-1}) \cap (0, 1); \tag{28}$$

$$\pi_h^B(q) < \tilde{\pi}_\ell^B(q_0, \bar{\delta}, v_L^{k-1}(q'), v_B^{k-1}(q')). \tag{29}$$

Condition (27) represents the same fixed point problem as in (22): if agents expect their continuation payoffs to be that of a $(k-1)$-step equilibrium in which the initial fraction of type H sellers is q', then aggregate behavior must be such that the fraction of type H sellers in the market after one period of trade is indeed q'. Crucially, for the analysis with restricted price offers, the mapping described in (22) was shown to be single-valued, continuous, and strictly increasing in q_0. Establishing these properties for the mapping described in (27) would allow for the same clean characterization of equilibria with unrestricted price offers. Unfortunately, we cannot verify these properties analytically; the law of motion (26) depends explicitly on buyers' offers, which in turn depend on future payoffs (and these depend on q' and future offers), which makes the analysis considerably more complex.

However, it is straightforward to use the recursive characterization provided by conditions (27) to (29) to compute the equilibrium set numerically. As an example, let $u_H = 1$, $c_H = 0.6$, $u_L = 0.3$, and assume δ is drawn from a uniform distribution with $\underline{\delta} = 0.2$ and $\bar{\delta} = 0.9$.[26] This numerical exercise (and many others like it) helps to confirm that the restriction to two prices is not driving the qualitative features of the equilibrium set. First, the mapping described in (27) is

[26] While there is nothing particularly special about our choice of values for u_H, c_H, u_L, $\underline{\delta}$, and $\bar{\delta}$, the restriction to a uniform distribution is helpful in that it ensures a unique solution to (6).

Table 1
Equilibrium regions with unrestricted price offers.

$\underline{q}^4$	$\underline{q}^3$	$\underline{q}^2$	$\underline{q}^1$	$\bar{q}^4$	$\underline{q}^0$	$\bar{q}^3$	$\bar{q}^2$	$\bar{q}^1$
0	0.008	0.254	0.445	0.499	0.503	0.547	0.579	0.603

indeed strictly increasing in q_0. Thus, there is a sequence of cutoffs—constructed in a manner analogous to that of Proposition 2—that partitions the interval (0, 1) in the same way as in the case with restricted price offers. Table 1 reports these cutoffs (up to 4-step equilibria). Second, the equilibrium regions overlap, so that there is multiplicity of equilibria, and it can be substantial; in the example, 0-, 1-, 2-, and 3-step equilibria coexist when $q_0 \in (0.503, 0.547)$. Third, the cut-offs deliver the same monotonicity found in the case of restricted price offers: as q_0 gets smaller, the minimum (or maximum) number of periods it takes for the market to clear increases. Finally, focusing on any particular equilibrium, the trading dynamics have the same qualitative features as in the environment with restricted price offers. For example, Fig. 3 plots average price offers, p_t^{avg}, and the fraction of high-quality assets, q_t, in a 3-step equilibrium with $q_0 = 0.1$. Both variables exhibit the same patterns as they do in the benchmark model with two prices.

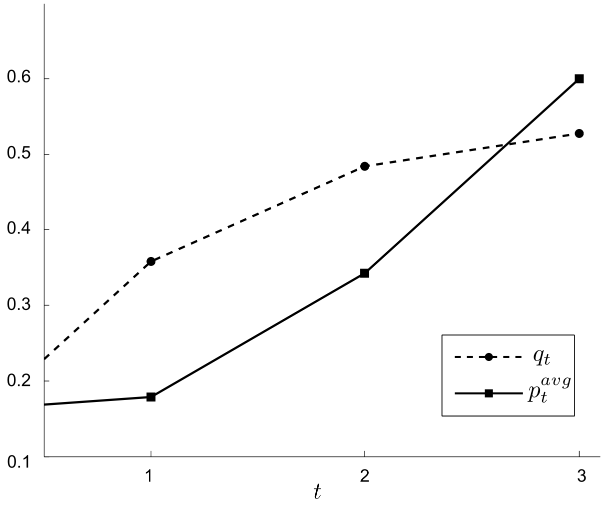

Fig. 3. The dynamics of trade.

5. Application: intervention in frozen markets

Frozen markets can cause serious damage to an economy. For example, during the recent financial crisis the market for asset-backed securities seized up, crippling the financial sector and ultimately producing a contraction in the provision of credit that has had long-lasting effects on the overall economy. Many believe that an important factor underlying this market freeze was asymmetric information about the quality of the assets for sale.[27] Therefore, a natural question

[27] The financial institutions that were selling these assets often had analysts that had purchased the underlying assets (e.g., mortgages), studied their properties, and worked closely with the rating agencies to bundle them into more opaque

emerges: can a government (or central bank) intervene to rejuvenate a market suffering from adverse selection?

Since our framework describes explicitly how markets can recover over time on their own, it also provides a natural setting to analyze how policies aimed at restoring liquidity can speed up (or slow down) this process. In this section, we analyze a particular program that was proposed in the US in March of 2009 with an explicit objective of restoring trade in the frozen market for asset-backed securities. We extend our model to capture the essential features of this program, and illustrate how our environment can provide unique—and perhaps counter-intuitive—insights into the efficacy of such policy interventions.

The policy The policy we analyze was called the Public–Private Investment Program for Legacy Assets, or "PPIP." Under this program, the government issued non-recourse loans to private investors to assist in buying legacy assets, with a minimum fraction of the purchase price being financed by the investor's own equity. This program essentially subsidizes a buyer's purchase by partially insuring his downside loss; if the asset turns out to be a lemon, the buyer can default and incur only a fraction of the total loss from the purchase, i.e., his equity investment.[28] Intuitively, such a program would seem to unambiguously (at least weakly) improve market liquidity. Indeed, in the context of a static or stationary model, any policy that reduces the loss to a buyer of acquiring a lemon should raise equilibrium prices, and thus increase the incentive for high-quality sellers to trade. The only question would seem to be about the optimal size of such an intervention.

However, within the context of a non-stationary environment, we show that such a program can potentially *slow down* market recovery. The crucial insight is that an intervention of this type potentially increases both current and future payoffs to a low-quality seller. If the increase in future payoffs is large relative to the increase in current payoffs, then a low-quality seller can in fact become less willing to trade immediately, thus slowing down the thawing process described in our equilibrium characterization.

Extending the model In order to analyze the policy described above, suppose now that a buyer who pays price p for an asset can borrow $(1-\gamma)p$ from the government. For simplicity, assume the buyer observes the quality of the asset before deciding whether to pay back the loan or default on the loan and surrender the asset. In this case, a buyer who pays price p for an asset of quality j repays his loan if, and only if, $u_j - (1-\gamma)p > 0$. Thus, a buyer who receives a high-quality asset always repays his loan, as does a buyer who pays $p < u_L$ for a low-quality asset. However, a buyer who pays $p_h = c_H$ for a low-quality asset defaults if $\gamma \leq 1 - u_L/p_h$. So, this policy amounts to a transfer $\tau = (1-\gamma)p_h - u_L \in [0, p_h - u_L]$ to the buyers who pay p_h for a low-quality asset.

final products. An extreme example of this asymmetric information is the "Abacus" deal, in which Goldman Sachs created and sold collateralized debt obligations to investors while simultaneously betting against them. In general, there are many reasons to believe that financial institutions have better information about the quality of their assets than potential buyers, perhaps because they learn about the asset while they own it (Bolton et al. [4]), or because they conduct research about the asset in anticipation of selling it (Guerrieri and Shimer [22]). By now, there is a large literature on the role of asymmetric information in the financial crisis; see, e.g., Gorton [21].

[28] See http://www.treasury.gov/initiatives/financial-stability/TARP-Programs/credit-market-programs/ppip/Pages/default.aspx for details on PPIP.

Analysis In order to understand the effects of providing buyers with partial insurance against acquiring a lemon, we first examine how the transfers described above change the structure of the equilibrium set in the model with restricted price offers. To this end, let

$$\pi_h^B(q_0,\tau) = q_0(u_H - p_h) + (1-q_0)(u_L - p_h + \tau)$$

denote the payoff to a buyer from offering $p_h = c_H$ given a transfer τ; sellers still accept any price offer $p \geq c_H$ in equilibrium. The payoff to a buyer from offering p_ℓ is the same as before, and the characterization of the equilibrium set proceeds in exactly the same way as in Section 4. In particular, Proposition 2 is still valid with the only difference being that, in the recursive procedure that determines equilibrium payoffs, the payoff to a buyer in a 0-step equilibrium is now $\pi_h^B(q_0,\tau)$; the payoff to a type L seller in a 0-step equilibrium is still $v_L^0 \equiv p_h$.

Let $v_B^k(q_0,\tau)$ and $v_L^k(q_0,\tau)$ be, respectively, the payoffs to buyers and type L sellers in a k-step equilibrium when the transfer is τ. Moreover, let $\underline{q}^k(\tau)$ and $\bar{q}^k(\tau)$ be, respectively, the lower and upper cutoffs for a k-step equilibrium given τ. The cutoff $\underline{q}^0(\tau)$ is the unique value of q_0 such that

$$\pi_h^B(q_0,\tau) = \pi_\ell^B\big(q_0,\bar{\delta},v_L^0,v_B^0(q_0,\tau)\big),$$

while $\underline{q}^1(\tau)$ and $\bar{q}^1(\tau)$ are the only values of q_0 satisfying the following two equations, respectively:

$$q^+\big(q_0,v_L^0\big) = \underline{q}^0(\tau);$$
$$\pi_\ell^B\big(q_0,\bar{\delta},v_L^0,v_B^0\big(q^+\big(q_0,v_L^0\big),\tau\big)\big) = \pi_h^B(q_0,\tau).$$

It is straightforward to show that $\underline{q}^0(\tau)$, $\underline{q}^1(\tau)$, and $\bar{q}^1(\tau)$ are decreasing in τ. Therefore, if the initial fraction of high-quality assets is sufficiently large, then an increase in τ can decrease the amount of time it takes for the market to clear. Intuitively, since the transfer τ increases the payoff from offering p_h, buyers are more willing to offer the high price.

The policy under consideration has a second, opposing effect, though. Since buyers are more willing to offer p_h when they are partially insured against buying a lemon, the average price sellers receive in the future increases as τ grows larger. Ceteris paribus, this makes sellers more likely to reject offers of p_ℓ in early rounds of trade, opting instead to wait for larger future payoffs. To see this, let $\xi^1(q_0,\tau)$ be the mass of buyers who offer p_h in the first period of trade in a 1-step equilibrium when the transfer is τ. The payoff to a type L seller in a 1-step equilibrium is then

$$v_L^1(q_0,\tau) = \xi^1(q_0,\tau)p_h + \big(1-\xi^1(q_0,\tau)\big)\int \max\big\{p_\ell,\delta v_L^0\big\}dF(\delta).$$

Straightforward algebra shows that $\xi^1(q_0,\tau)$, and thus $v_L^1(q_0,\tau)$, are increasing in τ.

Now observe that $\underline{q}^2(\tau)$ satisfies

$$q^+\big(\underline{q}^2(\tau),v_L^1\big(\underline{q}^1(\tau),\tau\big)\big) = \underline{q}^1(\tau). \tag{30}$$

Thus, as τ increases, two opposing forces are at work. On the one hand, since $\underline{q}^1$ is decreasing in τ, this tends to decrease $\underline{q}^2$ as well; holding v_L^1 constant in (30), $\underline{q}^2$ is decreasing in $\underline{q}^1$. On the other hand, holding $\underline{q}^1$ fixed, v_L^1 is increasing in τ, which makes sellers more likely to reject an offer of p_ℓ at $t=0$. This implies a smaller increase in the fraction of high-quality assets, and hence a larger value of $\underline{q}^2$. The second effect is not present in a 1-step equilibrium since v_L^0

Table 2
The effects of PPIP on the set of equilibrium.

Policy	$\underline{q}^4$	$\underline{q}^3$	$\underline{q}^2$	$\underline{q}^1$	$\bar{q}^4$	$\underline{q}^0$	$\bar{q}^3$	$\bar{q}^2$	$\bar{q}^1$
$\tau = 0$	0	0.008	0.254	0.445	0.499	0.503	0.547	0.579	0.603
$\tau = 0.5(c_H - u_L)$	0	0.064	0.238	0.341	0.385	0.381	0.427	0.459	0.485

is constant in τ, which explains why $\underline{q}^1$ is unambiguously decreasing in τ. However, when the market is two or more periods away from clearing, which is typically the case when the fraction of high-quality assets is small, the second effect is active, and can even dominate the first effect. In other words, when the lemons problem is severe, subsidizing the purchase of assets can increase the time required for market clearing, thus making the market *less liquid*.

Unrestricted offers The tension described above does not depend on the assumption that a buyer can offer only one of two prices. Indeed, the main effect of the policy in question is to change the *probability* that a buyer makes a high offer instead of a low offer; the effect on the low offer itself is second order. We confirm this by returning to the numerical example from Section 4.

Table 2 summarizes the effect of a transfer τ that is equal to 50% of the loss from purchasing a lemon at price c_H; we obtain similar results for different values of τ. One can see that in this case, the transfer allows markets to clear faster if the initial fraction of high-quality assets is large, but it has little effect on—and can even increase—the time to market clearing if the initial fraction of high-quality assets is small. Consider, for example, an economy with $q_0 = 0.4$. With the transfer in place, there exists an equilibrium in which the market clears immediately, while it takes until at least $t = 2$ for the market to clear in the absence of the transfer. However, the opposite is true for, say, $q_0 = 0.05$: the transfer increases the minimum number of periods before the market clears from three to four.[29]

These results suggest that the effects of an intervention in a dynamic lemons market depend crucially on both the size of the intervention (i.e., τ) *and its duration*. In particular, given the potentially damaging effects highlighted in Table 2, policymakers will typically want to intervene in a frozen market immediately and then implement a "sunset clause," in which the policy expires after a pre-specified amount of time. Such a policy can capitalize on the beneficial effects of providing buyers with the incentive to offer a higher price, while limiting the deleterious effects of raising low-quality sellers' future payoffs, which can cause them to delay trade.

Of course, a proper comparison of various policies also requires taking into account the costs of government funding. Once these costs are introduced, an interesting trade-off arises. By subsidizing buyers' purchases in early trading periods, one provides buyers with the incentive to trade early, which is good for welfare as gains from trade are realized with minimal delay. However, many of these trades will involve low-quality assets, which will be costly to subsidize. Alternatively, if the policy is stretched out over a longer horizon, trade occurs more slowly, but the market mechanism ensures that few low-quality assets are sold at high prices, which economizes on the policy's cost. Hence, in general, the optimal policy will involve a subsidy for some positive, but finite amount of time. While a full analysis of this trade-off is beyond the scope of the current paper, Fuchs and Skrzypacz [17] study this problem in a similar environment and characterize the optimal duration of interventions in dynamic markets suffering from adverse selection.

29 Since the effect of transfers on the equilibrium cutoffs is ambiguous, the same is true for the effect of transfers on market liquidity.

6. The assumption of one-time entry

The main goal of this paper is to analyze trading patterns in dynamic, decentralized markets suffering from adverse selection, with a particular focus on understanding how these markets can potentially recover on their own. Given the well-known challenges of characterizing equilibria in non-stationary environments, we imposed several restrictions on our model economy in order to retain tractability and illustrate our key insights in the clearest possible manner. In this section, we highlight the assumption that buyers and sellers exit the market after trading and discuss how our results might change if this assumption were relaxed.

Suppose a buyer who has acquired an asset is allowed to re-enter the market as a seller after learning the quality of the asset he has purchased. Clearly, a buyer who learns that he has acquired a high-quality asset has no incentive to re-enter. A buyer who learns that he has acquired a low-quality asset, however, might like to re-enter in hope of selling this asset to an unsuspecting buyer at a price $p > u_L$. A natural question is whether buyers in this situation would indeed re-enter the market if they were allowed and, if so, how this would affect equilibrium outcomes.

The precise answer to this question depends heavily on the structure of the market in which the asset is traded. For example, one has to specify exactly what type of information is available about the history of transactions for a particular agent or asset. In an environment with perfect record-keeping, the decision to re-sell an asset soon after acquiring it could reveal that the asset is of low quality, thus eliminating any gains from trade and rendering the option to re-enter moot.[30] One also has to specify how long it takes for a buyer to discover the quality of the asset he has acquired. Indeed, it might be the case that a buyer learns that he has obtained a low-quality asset only after the market has cleared, again rendering the option for re-entry irrelevant.[31] However, even if buyers learn the quality of their asset immediately and are able to re-enter the market anonymously, there are good reasons to believe that the basic insights that emerge from our benchmark model will be preserved. We sketch the argument here; a full-blown analysis of the model with re-sale is beyond the scope of this paper.

Suppose buyers learn the quality of their asset immediately after trade and then have the option to re-enter the market anonymously; i.e., should they choose to re-enter the market, they are indistinguishable from other sellers. Note first that if the market does not clear at $t = 0$, then at least a small fraction of buyers who have acquired a low-quality asset will definitely re-enter the market at some $t \geq 1$. If this were not the case, then the market would clear at some $1 \leq T < \infty$ (by Proposition 1), but then an individual buyer who acquired a low-quality asset in period $T - 1$ has incentive to re-enter the market in period T. Hence, some re-entry will occur and this has the potential to slow down a recovery.

However, even though the process might take longer, *markets will still thaw*, in the sense that all high-quality assets will be sold in a finite number of periods with probability one. The argument is as follows. In any period t' in which there are a positive measure of agents in the market,

[30] Indeed, information about past trades is available in a variety of markets. In some cases there are formal records of past transactions; for example, Carfax provides a history report for used vehicles. In other cases information about past trades is shared informally; for example, traders at investment banks typically find out who ended up purchasing a large pool of asset-backed securities through word-of-mouth. Kim [29] studies a dynamic lemons market in which information about sellers' past behavior is available and shows that equilibrium outcomes can be quite sensitive to the exact nature of the information available.

[31] There are also a number of markets in which the question of re-sale does not arise because of the properties of the good being traded. These include markets for non-durable or non-transferable goods (e.g., labor services), as well as markets where the cost of re-sale is fairly large (e.g., housing).

it cannot be the case that high-quality assets never trade again. Suppose, towards a contradiction, that only low-quality assets are traded in all periods $t \geq t'$. Then it must be that prices never exceed u_L in all periods $t \geq t'$, in which case buyers would have never have incentive to re-enter the market. However, in the absence of re-entry, Proposition 1 shows that all high-quality assets are eventually sold. Therefore, whenever a positive measure of agents remain in the market, it must be the case that high-quality assets are traded in future periods with strictly positive probability. Moreover, since buyers discount the future, one can show that this probability must be strictly bounded away from zero.[32] An implication of this last result is that the probability of a high-quality asset being sold in a finite number of periods is one. Thus, our result that frozen markets suffering from adverse selection thaw over time endogenously survives when re-entry is possible.

7. Conclusion

This paper provides a theory of how markets suffering from adverse selection can recover over time on their own. Sellers with low-quality assets exit the market relatively more quickly than those with high-quality assets, causing the average quality of assets in the market to increase over time. Eventually, all assets are exchanged. The model delivers sharp predictions about how long this process takes, or the extent to which the market is illiquid, as well as the behavior of prices over time. Interestingly, we find multiple equilibria, which suggests that there is scope for coordination failures in dynamic, decentralized markets with adverse selection. We argue that this model serves as a useful benchmark for understanding how exogenous events or interventions will affect the speed with which markets recover. We provide a specific example from the recent financial crisis, and show how accounting for dynamic considerations can shed light on potentially harmful, unintended effects of policies aimed at restoring liquidity in frozen markets.

Our model is intentionally stylized in order to retain tractability. However, we believe that the insights that emerge from our analysis are robust to a number of perturbations to the economic environment. In Section 6, we argued that the assumption of one-time entry was not necessarily crucial for many of our results. One can also show that the qualitative properties of our equilibrium characterization are preserved under alternative methods for calculating payoffs when there is a zero measure of agents remaining in the market. Finally, we conjecture that allowing buyers to offer sellers a menu of contracts, including lotteries, would not significantly alter the dynamics of trade. Intuitively, buyers would attempt to separate sellers with different quality assets by offering high prices with a low probability of trade, and low prices with a high probability of trade.[33] Hence, as in our model, low-quality assets would exit the market at a faster rate than high-quality assets and the lemons problem would dissipate over time.

Other natural extensions to our model include allowing sellers the choice of what type of asset to generate and when to enter the market, allowing buyers to acquire costly information about an asset's quality, and introducing aggregate uncertainty and learning. These are left for future work.

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[32] To see this, let $k \geq 2$ be the smallest integer such that $\bar{\delta}^{k-1} c_H \leq u_L$, and let $\eta > 0$ satisfy $\eta c_H + (1-\eta)\bar{\delta}^k c_H = u_L$. Now, consider a period $t' \geq 1$ in which re-entry occurs. The probability that a seller in the market in period t' receives an offer of c_H any time between periods t' and $t' + k$ must be at least η. Otherwise, by construction, the payoff from re-entry in period t' is smaller than $\eta c_H + (1-\eta)\bar{\delta}^k c_H = u_L$ and hence re-entry is not profitable.

[33] This is the basic idea in Guerrieri and Shimer [22].

Appendix A. Omitted proofs and results

Proof of Lemma 1

Suppose σ^* is an equilibrium. First note that no buyer offers more than u_H. Hence, sellers accept any price offer $p \geq p^* = \max\{c_H, \bar{\delta}u_H\}$. This, however, implies that no buyer offers $p > p^*$. Now define $\{p_n\}_{n=1}^{\infty}$ to be such that $p_1 = p^*$ and $p_{n+1} = \max\{c_H, \bar{\delta}p_n\}$ for all $n \geq 1$. Clearly, if no buyer offers more than p_n, then sellers accept any price offer $p \geq p_{n+1}$. This implies that no buyer offers more than p_{n+1}. The desired result now follows from the fact that $\{p_n\}$ converges to c_H.

Lemma 3 and proof

Here, we state and prove Lemma 3 without assuming that problem (6) has a unique solution. Notice that if $u_L - \underline{\delta}v_L - \delta v_B \leq 0$, then any $p \leq \underline{\delta}v_L$ is an optimal solution to (6). In what follows, we abstract from this trivial source of multiplicity and assume that the optimal solution to (6) when $u_L - \underline{\delta}v_L - \delta v_B \leq 0$ is $p = \underline{\delta}v_L$.

The result below, which follows from (3), is necessary for the proof of Lemma 3.

Lemma 9. *If σ^* is an equilibrium and $T(\sigma^*) < \infty$, then $V^B_{T(\sigma^*)}(\sigma^*) > 0$.*

Proof. Let σ^* be an equilibrium with $T(\sigma^*) = T < \infty$. We claim that $q_T[u_H - c_H] + (1 - q_T)[u_L - c_H] > 0$. Suppose not. Since the highest continuation payoff possible to a seller who does not trade in period T is c_H, assumption (3) implies that a type L seller in the market in period T accepts any price offer $p \in (\underline{\delta}c_H, u_L)$ with positive probability. Hence, any buyer in the market in period T obtains a positive payoff if he offers $p \in (\underline{\delta}c_H, u_L)$, a contradiction. □

Lemma 3. *Let σ^* be an equilibrium. In any period $t \leq T(\sigma^*)$, there exists $\hat{\delta}_t \in [\underline{\delta}, \bar{\delta}]$ such that a buyer offers $p = c_H$ if, and only if, $\delta \leq \hat{\delta}_t$. Moreover, for all $\delta' > \delta > \hat{\delta}_t$, if p' and p are optimal price offers for a buyer with discount factor δ' and δ, respectively, then $p \geq p'$.*

Proof. Let σ^* be an equilibrium. We know from Proposition 1 that $T(\sigma^*) < \infty$. The desired result holds trivially in period $T(\sigma^*)$, when all buyers in the market offer $p = c_H$. Let then $t < T(\sigma^*)$ and suppose that: (*i*) the fraction of high-quality assets in the market in period t is q; and (*ii*) the continuation payoffs to buyers and type L sellers who do not trade in period t are v_B and v_L, respectively. Note that v_B and v_L are positive, as any agent has the option of only trading in period $T(\sigma^*)$ and (by Lemma 9) the payoff to a buyer in the last period of trade is positive. Now let

$$\pi^B_\ell(p, \delta) = (1 - q)F\left(\frac{p}{v_L}\right)[u_L - p] + \left\{q + (1 - q)\left[1 - F\left(\frac{p}{v_L}\right)\right]\right\}\delta v_B$$

be the payoff to a buyer with discount factor δ who offers $p < c_H$ in period t, and define $\pi^B_\ell(\delta)$ and $\Psi^B_\ell(\delta)$ to be such that $\pi^B_\ell(\delta) = \max_{p<c_H} \pi^B_\ell(p, \delta)$ and $\Psi^B_\ell(\delta) = \operatorname{argmax}_{p<c_H} \pi^B_\ell(p, \delta)$; both $\pi^B_\ell(\delta)$ and $\Psi^B_\ell(\delta)$ are well-defined since it is not optimal for a buyer to offer $\min\{\bar{\delta}v_L, u_L\} < p < c_H$ in period t. A straightforward argument shows that $\pi^B_\ell(\delta)$ is strictly increasing in δ. Moreover, since

$$\frac{\partial^2 \pi^B_\ell}{\partial\delta\partial p}(p, \delta) = -\frac{v_B}{v_L}F'\left(\frac{p}{v_L}\right) < 0$$

for all $p > \underline{\delta} v_L$, Theorem 1 in Edlin and Shannon [14] shows that if $p' \in \Psi_\ell^B(\delta')$ is such that $\underline{\delta} v_L < p' < \min\{\bar{\delta} v_L, c_H\}$, then $p > p'$ for all $p \in \Psi_\ell^B(\delta)$ with $\delta < \delta'$. The desired result follows from the fact that one of the following three alternatives hold: $\pi_\ell^B(\delta) > \pi_h^B = q u_H + (1-q) u_L - c_H$ for all $\delta \in [\underline{\delta}, \bar{\delta}]$, $\pi_\ell^B(\delta) < \pi_h^B$ for all $\delta \in [\underline{\delta}, \bar{\delta}]$, or there is a unique $\hat{\delta} \in [\underline{\delta}, \bar{\delta}]$ with $\pi_\ell^B(\hat{\delta}) = \pi_h^B$. □

Proof of Proposition 1

Suppose, by contradiction, that there exists an equilibrium σ^* in which the market never clears. For each $t \geq 0$, let: (*i*) q_t be the fraction of high-quality assets in the market in period t; (*ii*) $v_t^L > 0$ be the (ex-ante) payoff to a type L seller in the market in period t; (*iii*) v_t^B be the payoff to a buyer in the market in period t; and (*iv*) $p_t(\delta)$ be the price a buyer in the market in period t offers if his discount factor is δ.[34]

Now let q^* be such that $q^* u_H + (1-q^*) u_L = c_H$ and $q_\infty = \lim_{t\to\infty} q_t$. We claim that $q_\infty < 1$. Indeed, let q^{**} be such that $q^{**} u_H + (1-q^{**}) u_L - c_H = \pi^{**}$, where $\pi^{**} = \max\{u_L, \bar{\delta}(u_H - c_H)\}$, and suppose that the fraction of high-quality assets in the market is greater than q^{**}; $q^{**} < 1$ since $u_H - c_H > u_L$. Then, a buyer's payoff if he offers $p = c_H$ is greater than π^{**}. On the other hand, since $u_H - c_H$ is the greatest payoff possible to a buyer, the payoff to a buyer with discount factor δ who offers $p < c_H$ is bounded above by $\bar{\pi}(\delta) = \max\{u_L - p, \delta(u_H - c_H)\}$. Since $\bar{\pi}(\delta) \leq \pi^{**}$, we can then conclude that the market clears immediately. Thus, $q_\infty \leq q^{**}$.

There are three cases to consider: (*1*) $q_\infty \leq q^*$ and $q_t < q_\infty$ for all $t \geq 0$; (2) $q_\infty \leq q^*$ and $q_t = q_\infty$ after finitely many periods; and (*3*) $q_\infty > q^*$. We consider these three cases in turn.

(*1*) Since $q_t < q_\infty \leq q^*$ for all $t \geq 0$, it is never optimal for a buyer to offer c_H. Hence, the mass of type H sellers who trade in each period is zero. Now let $A_t^L(p) = F(p/v_{t+1}^L)$ be the probability that a type L seller accepts an offer $p \in [0, c_H)$ in period t. We can write the law of motion (4) as

$$q_{t+1} = \frac{q_t}{q_t + (1-q_t)(1-\chi_t)},$$

where $\chi_t = \int_{\underline{\delta}}^{\bar{\delta}} A_t^L(p_t(\delta)) dF(\delta)$. First, note that $q_\infty < 1$ only if $\{\chi_t\}$ converges to zero. Now, observe that $\lim_{t\to\infty} \chi_t = 0$ implies that both $\{v_t^L\}$ and $\{v_t^B\}$ converge to zero, as χ_t is the ex-ante probability of trade in period t. Finally, note that since $\lim_{t\to\infty} v_t^L = 0$, there exists $t_0 \geq 0$ such that if $t \geq t_0$, then the probability that an offer of $u_L/2$ is accepted by a type L seller is at least 1/2. Thus, $t \geq t_0$ implies that $v_t^B \geq (1-q_t) u_L/4$, and so $\lim_{t\to\infty} v_t^B > 0$, a contradiction.

(*2*) First, note that the same argument as above rules out the case in which $q_\infty < q^*$. Suppose then that $q_\infty = q^*$ and there exists $t_0 \geq 0$ such that $q_t = q_\infty$ for all $t \geq t_0$. This implies that in every period $t \geq t_0$, buyers in the market either offer $p = c_H$ or make an offer $p < c_H$ that is rejected by all type L sellers. This, however, implies that $v_t^B = 0$ for all $t \geq t_0$, in which case the proof of Lemma 9 shows that buyers can profitably deviate by offering $p \in (\underline{\delta} c_H, u_L)$.

(*3*) Assume, without loss of generality, that $q_0 > q^*$. Now let $\pi_h^B(q) = q u_H + (1-q) u_L - c_H$ and

$$\pi_{\ell,t}^B(\delta) = \max_{p < c_H} (1-q_t) A_t^L(p)[u_L - p] + \{q + (1-q)[1 - A_t^L(p)]\} \delta v_{t+1}^B.$$

[34] To see that $v_t^L > 0$ for all $t \geq 0$, note that $v_t^L = 0$ requires that all buyers in the market make offers that are rejected by all type L sellers in every period $t' \geq t$. This, however, implies that a buyer in the market in period t can profitably deviate by offering a small positive price.

Since $v_t^B > 0$ for all $t \geq 0$, $\pi_{\ell,t}^B(\delta)$ is strictly increasing in δ. Moreover, $\pi_{\ell,t}^B(\bar{\delta}) > \pi_h^B(q_t)$ for all $t \geq 0$, otherwise the market clears in finite time. So, for all $t \geq 0$, there exists a unique $\hat{\delta}_t \in [\underline{\delta}, \bar{\delta})$ such that an offer of $p = c_H$ in period t is strictly optimal if, and only if, $\delta < \hat{\delta}_t$. Let then $\widetilde{\chi}_t = \int_{\hat{\delta}_t}^{\bar{\delta}} A_t^L(p_t(\delta))dF(\delta)$ and $\gamma_t = [1 - F(\hat{\delta}_t)]^{-1}\widetilde{\chi}_t$. We can write the law of motion (4) as

$$q_{t+1} = \frac{q_t}{q_t + (1-q_t)[1-\gamma_t]}.$$

Since $\pi_{\ell,t}^B(\delta)$ is continuous in δ, it is easy to show that $\{\hat{\delta}_t\}$ is a convergent sequence. Denote its limit by $\hat{\delta}_\infty$. There are two possibilities: $\hat{\delta}_\infty < \bar{\delta}$ or $\hat{\delta}_\infty = \bar{\delta}$.

(*3.1*) Suppose that $\hat{\delta}_\infty < \bar{\delta}$. As in the first case, $q_\infty < 1$ only if $\{\gamma_t\}$ converges to zero. This, however, is only possible if $\{\widetilde{\chi}_t\}$ converges to zero, in which case $\{v_t^B\}$ converges to $v_\infty^B < \pi_h^B(q_\infty)$; note that $\widetilde{\chi}_t$ is the ex-ante probability that a buyer transacts in period t conditional on the event that he makes an offer $p < c_H$ and $\hat{\delta}_\infty < \bar{\delta}$ implies that in every period t the probability that a buyer in the market offers $p < c_H$ is bounded away from zero. On the other hand, an option for a buyer in the market in period t is to offer $p = c_H$ regardless of his discount factor. This implies that $v_t^B \geq \pi_h^B(q_t)$ for all $t \geq 0$, which, in turn, implies that $v_\infty^B \geq \pi_h^B(q_\infty)$, a contradiction.

(*3.2*) Suppose now that $\hat{\delta}_\infty = \bar{\delta}$. In this case, $\{v_t^B\}$ converges to $\pi_h^B(q_\infty)$. Let $\varepsilon > 0$ be such that $\varepsilon u_L < (1-\bar{\delta})\pi_h^B(q_\infty)/4$. Moreover, let $t_0 \geq 0$ be such that $v_t^B \leq (1+\bar{\delta})\pi_h^B(q_\infty)/2\bar{\delta}$ and $(3+\bar{\delta})\pi_h^B(q_\infty)/4 < \pi_h^B(q_t)$ for all $t \geq t_0$. By construction, $t \geq t_0$ and $A_t^L(p) \leq \varepsilon$ imply that

$$(1-q_t)A_t^L(p)[u_L - p] + \{q_t + (1-q_t)[1 - A_t^L(p)]\}\delta v_{t+1}^B < \pi_h^B(q_t).$$

Thus, when $t \geq t_0$, any price offer $p \in [0, c_H)$ such that $A_t^L(p) \leq \varepsilon$ is not optimal for a buyer regardless of his discount factor. Now observe, as above, that $q_\infty < 1$ only if $\{\gamma_t\}$ converges to zero. Suppose that $\{\gamma_t\}$ converges to zero and let $t \geq t_0$ be such that $\gamma_t < \varepsilon$. Then, there exists an open set $S \subseteq (\hat{\delta}_t, \bar{\delta})$ such that $A_t^L(p_t(\delta)) < \varepsilon$ for all $\delta \in S$. Consequently, when t is large, a positive mass of buyers behaves sub-optimally, a contradiction. This completes the proof.

Proof of Lemma 6

Let $\eta^0(q,\delta) = \pi_h^B(q) - \pi_\ell^B(q,\delta,v_L^0,v_B^0(q))$. Since $u_L - p_\ell \geq \bar{\delta}(u_H - p_h) \geq \bar{\delta}v_B^0(q)$, straightforward algebra shows that $\partial\eta^0(q,\delta)/\partial q > 0$. Now note that $\eta^0(0,\bar{\delta}) < 0 < \eta^0(1,\bar{\delta})$. Given that $\eta^0(q,\delta)$ is continuous in q, there exists a unique $\underline{q}^0 \in (0,1)$ such that $\eta^0(q_0,\bar{\delta}) \geq 0$ if, and only if, $q_0 \geq \underline{q}^0$. Moreover, $\eta(\underline{q}^0,\bar{\delta}) = 0$ implies that

$$v_B^0(\underline{q}^0)\left[1 - \bar{\delta} + (1-\underline{q}^0)F\left(\frac{p_\ell}{p_h}\right)\bar{\delta}\right] = (1-\underline{q}^0)F\left(\frac{p_\ell}{p_h}\right)[u_L - p_\ell],$$

and so $v_B^0(\underline{q}^0) > 0$. Thus, σ^0 is an equilibrium if, and only if, $q_0 \in [\underline{q}^0, 1)$.

Suppose now that $q_0 < \underline{q}^0$ and consider a strategy profile $\widetilde{\sigma}^0$ such that all buyers offer p_h in $t = 0$. One alternative for a buyer is to offer p_h in every period regardless of his discount factor. Let $\widetilde{\mathbf{p}}$ denote this strategy. It must be that $V_t^B(\widetilde{\sigma}^0) \geq V_t^B(\widetilde{\mathbf{p}}|\widetilde{\sigma}^0)$ for all $t \geq 0$ if $\widetilde{\sigma}^0$ is to be an equilibrium. Now observe that if the probability of trade in each period is $\alpha \in (0,1)$, then

$$V_t^B(\widetilde{\mathbf{p}}|\widetilde{\sigma}^0,\alpha) = \sum_{\tau=1}^{\infty}\alpha(1-\alpha)^{\tau-1}(\mathbb{E}[\delta])^{\tau-1}v_B^0(q_{t+\tau-1}^\alpha)$$

for all $t \geq 0$, where $q_{t+\tau-1}^\alpha$ is the fraction of type H sellers in the market in period $t+\tau-1$. It is easy to see that the sequence $\{q_t^\alpha\}_{t=0}^\infty$ is non-decreasing, and so $v_B^0(q_{t+\tau-1}^\alpha) \geq v_B^0(q_0)$ for all $\tau \geq 1$. From this, it follows that $V_t^B(\widetilde{\mathbf{p}}|\widetilde{\sigma}^0) \geq v_B^0(q_0)$. Thus, $\widetilde{\sigma}^0$ is an equilibrium only if $V_1^B(\widetilde{\sigma}^0) \geq v_B^0(q_0)$. However, since $V_1^L(\widetilde{\sigma}^0) \leq p_h$ and $q_0 < \underline{q}^0$ implies that $\eta^0(q_0, \bar{\delta}) < 0$,

$$\pi_\ell^B\big(q_0, \bar{\delta}, V_1^L\big(\widetilde{\sigma}^0\big), V_1^B\big(\widetilde{\sigma}^0\big)\big) \geq \pi_\ell^B\big(q_0, \bar{\delta}, v_L^0, v_B^0(q_0)\big) > \pi_h^B(q_0)$$

for all $q_0 < \underline{q}^0$. Therefore, there exists $\delta' < \bar{\delta}$ such that it can *not* be optimal for a buyer with discount factor in $(\delta', \bar{\delta}]$ to offer p_h at $t = 0$, so that $\widetilde{\sigma}^0$ is not an equilibrium.

Proof of Lemma 7

Recall that $q^+(q, v_L^0)$ is strictly increasing in q when $p_\ell / v_L^0 < \bar{\delta}$ and that $q^+(q, v_L^0) \equiv 1$ otherwise. Hence, there exists a unique $\underline{q}^1 < \underline{q}^0$ such that $q^+(q_0, v_L^0) \geq \underline{q}^0$ if, and only if, $q_0 \in [\underline{q}^1, 1]$. Note that $\underline{q}^1 = 0$ if $p_\ell / v_L^0 \geq \bar{\delta}$ and $\underline{q}^1$ is such that $q^+(\underline{q}^1, v_L^0) = \underline{q}^0$ otherwise. Now let $\eta^1(q, \delta) = \pi_h^B(q) - \pi_\ell^B(q, \delta, v_L^0, v_B^0(q^+(q, v_L^0)))$. We claim that $\eta^1(q, \delta)$ is strictly increasing in q. Indeed,

$$\pi_\ell^B\big(q, \delta, v_L, \pi_h^B\big(q^+(q, v_L)\big)\big) = \delta \pi_h^B(q) + (1 - q) F\left(\frac{p_\ell}{v_L}\right)\big[u_L - p_\ell - \delta(u_L - p_h)\big] \quad (31)$$

for all $q \in (0, 1)$ and $\delta \in [\underline{\delta}, \bar{\delta}]$. Since $u_L < p_h$, a consequence of (31) is that if $q_0' > q_0$, then

$$\begin{aligned} &\pi_\ell^B\big(q_0', \delta, v_L, \pi_h^B\big(q^+\big(q_0', v_L\big)\big)\big) - \pi_\ell^B\big(q_0, \delta, v_L, \pi_h^B\big(q^+(q_0, v_L)\big)\big) \\ &\quad \leq \delta\big(q_0' - q_0\big)[u_H - u_L], \end{aligned} \quad (32)$$

from which the desired result follows. Given that $\eta^1(0, \bar{\delta}) < 0 < \eta^1(1, \bar{\delta})$ and $\eta^1(q, \delta)$ is continuous in q, there exists a unique $\bar{q}^1 \in (0, 1)$ such that $\eta^1(q_0, \bar{\delta}) < 0$ if, and only if $q_0 \in [0, \bar{q}^1)$. Hence, $\pi_h^B(q_0) < \pi_\ell^B(q_0, \bar{\delta}, v_L^0, v_B^0(q^+(q_0, v_L^0)))$ if, and only if $q_0 \in [0, \bar{q}^1)$. Next, observe that since $v_B^0(q^+(q_0, p_h)) > v_B^0(q_0)$ for all $q_0 \in (0, 1)$,

$$\pi_\ell^B\big(\underline{q}^0, \bar{\delta}, v_L^0, v_B^0\big(q^+\big(\underline{q}^0, p_h\big)\big)\big) > \pi_\ell^B\big(\underline{q}^0, \bar{\delta}, v_L^0, v_B^0\big(\underline{q}^0\big)\big) = \pi_h^B\big(\underline{q}^0\big).$$

Thus, $\eta^1(\underline{q}^0, \bar{\delta}) < 0$, from which we obtain that $\bar{q}^1 > \underline{q}^0$.

Lemma 10 and proof

Lemma 10. *The payoff v_B^1 is continuous in q_0 and $v_B^1(q_0') - v_B^1(q_0) \leq (q_0' - q_0)[u_H - u_L]$ for all $q_0' > q_0$. The fraction ξ^1 is continuous and increasing in q_0, with $\lim_{q_0 \to \bar{q}^1} \xi^1(q_0) = 1$. The payoff v_L^1 is continuous and increasing in q_0, with $\lim_{q_0 \to \bar{q}^1} v_L^1(q_0) = v_L^0$.*

Proof. Note that $q^+(q_0, v_L^0)$ continuous in q_0 implies that $v_B^1(q_0)$ is also continuous in q_0. We now prove that $v_B^1(q_0') - v_B^1(q_0) \leq (q_0' - q_0)[u_H - u_L]$ for all $q_0' > q_0$. Let $v_B^1(q_0, \delta)$ be the payoff to a buyer in a 1-step equilibrium when his discount factor is δ. Then

$$v_B^1(q_0, \delta) = \pi_\ell^B\big(q_0, \delta, v_L^0, v_B^0\big(Q_+^1(q_0)\big)\big) + \max\big\{\eta^1(q_0, \delta), 0\big\},$$

where $\eta^1(q, \delta) = \pi_h^B(q) - \pi_\ell^B(q, \delta, v_L^0, v_B^0(q^+(q, v_L^0)))$. Given that $v_B^1(q_0) = \int_{\underline{\delta}}^{\bar{\delta}} v_B^1(q_0, \delta) dF(\delta)$, we are done if we show that $q_0' > q_0$ implies that

$$v_B^1\big(q_0', \delta\big) - v_B^1(q_0, \delta) \leq \big(q_0' - q_0\big)[u_H - u_L] \quad (33)$$

regardless of δ. By the proof of Lemma 7, for each $\delta \in [\underline{\delta}, \bar{\delta}]$ there exists a unique $q^* = q^*(\delta) \in (0, 1)$ such that $\eta^1(q^*, \delta) \geq 0$ if, and only if, $q \geq q^*$; note that $q^*(\bar{\delta}) = \bar{q}^1$. Let then $q_0' > q_0$.

It follows immediately from (32) in the proof of Lemma 7 that (33) holds if $q_0' \leq q^*$. Since $v_B^1(q_0', \delta) - v_B^1(q_0, \delta) \leq v_B^1(q_0', \delta) - \pi_h^B(q_0)$, condition (33) also holds if $q_0' > q^*$.

Next, we prove that $\xi^1(q_0)$ is continuous and increasing in q_0, with $\lim_{q_0 \to \bar{q}^1} \xi^1(q_0) = 1$. Since $\pi_\ell^B(q_0, \delta, v_L^0, v_B^0(Q_+^1(q_0)))$ is strictly increasing in δ, $\eta^1(q, \delta)$ is strictly decreasing in δ. Let $\delta^1(q_0)$, be such that: *(i)* $\delta^1(q_0) = \underline{\delta}$ if $\eta^1(q_0, \underline{\delta}) \leq 0$; and *(ii)* $\eta^1(q_0, \delta^1(q_0)) = 0$ if $\eta^1(q_0, \underline{\delta}) > 0$. Given that $\eta^1(\bar{q}^1, \bar{\delta}) = 0$ and $\eta^1(q, \delta)$ is strictly increasing in q, $\delta^1(q_0)$ is uniquely defined for all $q_0 \in [\underline{q}^1, \bar{q}^1) \cap (0, 1)$. By construction, $\xi^1(q_0) = F(\delta^1(q_0))$. Since $\eta^1(q, \delta)$ is jointly continuous in q and δ, a standard argument shows that $\delta^1(q_0)$ is continuous in q_0. Moreover, $\delta^1(q_0)$ is strictly increasing in q_0 if $\eta^1(q_0, \underline{\delta}) > 0$, as $\eta^1(q, \delta)$ is strictly increasing in q. The desired result follows from the fact that $F(\delta)$ is continuous and strictly increasing in δ when $\delta \in [\underline{\delta}, \bar{\delta}]$ and $\lim_{q_0 \to \bar{q}^1} \delta^1(\bar{q}^1) = \bar{\delta}$.

Finally, note that the properties of $v_L^1(q_0)$ follow immediately from the properties of $\xi^1(q_0)$. □

Proof of Lemma 8

We first show that (19) and (20) imply (21). Suppose that $q' \in [\underline{q}^1, \bar{q}^1) \cap (0, 1)$. In order to prove that (21) is satisfied, it is sufficient to show

$$\pi_h^B(q') - \pi_h^B(q_0) \geq \pi_\ell^B\big(q', \delta, v_L^0, v_B^0\big(q^+\big(q', v_L^0\big)\big)\big) - \pi_\ell^B\big(q_0, \delta, v_L^1\big(q'\big), v_B^1\big(q'\big)\big) \tag{34}$$

for all $\delta \in [\underline{\delta}, \bar{\delta}]$. First observe that $\pi_\ell^B(q_0, \bar{\delta}, v_L^1(q'), v_B^1(q')) \geq \pi_\ell^B(q_0, \bar{\delta}, v_L^1(q'), \pi_h^B(q'))$ since $v_B^1(q') \geq \pi_h^B(q')$. Given that $q' = q^+(q_0, v_L^1(q'))$ by (19), Eq. (31) implies that

$$\begin{aligned}
&\pi_\ell^B\big(q', \bar{\delta}, v_L^0, v_B^0\big(q^+\big(q', v_L^0\big)\big)\big) - \pi_\ell^B\big(q_0, \bar{\delta}, v_L^1\big(q'\big), v_B^1\big(q'\big)\big) \\
&\quad \leq \bar{\delta}\big[\pi_h^B\big(q'\big) - \pi_h^B(q_0)\big] + \left\{\big(1 - q'\big)F\left(\frac{p_\ell}{v_L^0}\right) - (1 - q_0)F\left(\frac{p_\ell}{v_L^1(q')}\right)\right\} \\
&\qquad \times \big[u_L - p_\ell - \bar{\delta}(u_L - p_h)\big].
\end{aligned}$$

Since $v_L^0 > v_L^1(q')$ for all $q' \in [\underline{q}^1, \bar{q}^1) \cap (0, 1)$, $u_L < p_h$, and $q' > q_0$, the second term on the right side of the above inequality is negative, which confirms (34).

We now show that there exists a 2-step equilibrium if, and only if, $q_0 \in [\underline{q}^2, \bar{q}^2) \cap (0, 1)$. First note that since $\bar{q}^1 < 1$, (19) and (20) can be satisfied only if $p_\ell / v_L^1(q') < \bar{\delta}$. Now observe that if $p_\ell / v_L^1(q') < \bar{\delta}$, then

$$q^-\big(q'\big) = \frac{q'[1 - F(p_\ell / v_L^1(q'))]}{1 - q' F(p_\ell / v_L^1(q'))}$$

belongs to the interval (0, 1) and is such that $q^+(q^-(q'), v_L^1(q')) = q'$. Thus, (19) is satisfied for $q' \in [\underline{q}^1, \bar{q}^1) \cap (0, 1)$ if, and only if, $p_\ell / v_L^1(q') < \bar{\delta}$. Moreover, it is immediate to see that $q^-(q')$ is the only possible value of q_0 for which (19) and (20) can hold.

Since $v_L^1(q')$ is increasing in q', $p_\ell / v_L^1(\tilde{q}) < \bar{\delta}$ implies that $p_\ell / v_L^1(q') < \bar{\delta}$ for all $q' > \tilde{q}$. Let then $\tilde{q}^1$ be such that $\tilde{q}^1 = 0$ if $p_\ell / v_L^1(\underline{q}^1) < \bar{\delta}$ and $\tilde{q}^1 = \sup\{q' \in [\underline{q}^1, \bar{q}^1) : p_\ell / v_L^1(q') = \bar{\delta}\}$ otherwise; $\tilde{q}^1$ is well-defined since $p_\ell / v_L^1(\bar{q}^1) = p_\ell / v_L^0 < \bar{\delta}$. By construction, there exists $q_0 \in (0, 1)$ such that (19) and (20) are satisfied if, and only if, $q' \in [\underline{q}^1, \bar{q}^1) \cap (\tilde{q}^1, 1)$, in which case $q_0 = q^-(q')$. Now observe that $q^-(q')$ is continuous and strictly increasing in q'. Thus, $q^-(q')$ is invertible and its inverse $Q_+^2 : [\underline{q}^1, \bar{q}^1) \cap (\tilde{q}^1, 1) \to (0, 1)$ is continuous and strictly increasing.

By construction, we have that: (*i*) when $p_\ell/v_L^1(\underline{q}^1) < \bar{\delta}$, a 2-step equilibrium exists if, and only if, $Q_+^2(q_0) \in [\underline{q}^1, \bar{q}^1) \cap (0,1)$; (*ii*) when $p_\ell/v_L^1(\underline{q}^1) \geq \bar{\delta}$, a 2-step equilibrium exists if, and only if, $Q_+^2(q_0) \in (\widetilde{q}^1, \bar{q}^1)$. We are done if we show that $\lim_{q' \to \widetilde{q}^1} q^-(q') = 0$ when $p_\ell/v_L^1(\underline{q}^1) \geq \bar{\delta}$. This follows from the fact that $\lim_{q' \to \widetilde{q}^1} F(p_\ell/v_L^1(q')) = 1$.

Lemma 11 and proof

Lemma 11. *The payoff v_B^2 is continuous in q_0 and $v_B^2(q_0') - v_B^2(q_0) \leq (q_0' - q_0)[u_H - u_L]$ for all $q_0' > q_0$. The fraction ξ^2 is continuous and increasing in q_0, with $\lim_{q_0 \to \bar{q}^2} \xi^2(q_0) = \xi^1(\bar{q}^2)$. The payoff v_L^2 is continuous and increasing in q_0, with $v_L^2(\bar{q}^2) \equiv \lim_{q_0 \to \bar{q}^2} v_L^2(q_0) = v_L^1(\bar{q}^2)$ and $v_L^2(q_0) \leq v_L^1(Q_+^2(q_0))$ for all q_0.*

Proof. Since Q_+^2 is continuous in q_0, the continuity of v_B^2 follows from the continuity of v_B^1 and v_L^1. We now prove that if $q_0' > q_0$, then $v_B^2(q_0') - v_B^2(q_0) \leq (q_0' - q_0)[u_H - u_L]$. In order to do so, first note that if $q_0' > q_0$, then

$$
\begin{aligned}
&\pi_\ell^B\big(q_0', \delta, v_L^1\big(Q_+^2(q_0')\big), v_B^1\big(Q_+^2(q_0')\big)\big) - \pi_\ell^B\big(q_0, \delta, v_L^1\big(Q_+^2(q_0)\big), v_B^1\big(Q_+^2(q_0)\big)\big) \\
&= \left\{q_0' + (1-q_0')\left[1 - F\left(\frac{p_\ell}{v_L^1(Q_+^2(q_0'))}\right)\right]\right\}\delta\big[v_B^1\big(Q_+^2(q_0')\big) - v_B^1\big(Q_+^2(q_0)\big)\big] \\
&\quad + \big[\delta v_B^1\big(Q_+^2(q_0)\big) - (u_L - p_\ell)\big] \\
&\quad \times \left\{(1-q_0)F\left(\frac{p_\ell}{v_L^1(Q_+^2(q_0))}\right) - (1-q_0')F\left(\frac{p_\ell}{v_L^1(Q_+^2(q_0'))}\right)\right\} \\
&\leq \left\{q_0' + (1-q_0')\left[1 - F\left(\frac{p_\ell}{v_L^1(Q_+^2(q_0'))}\right)\right]\right\}\delta\big[Q_+^2(q_0') - Q_+^2(q_0)\big](u_H - u_L) \\
&\leq \delta\big(q_0' - q_0\big)[u_H - u_L].
\end{aligned} \tag{35}
$$

The first inequality follows from the fact that: (*i*) $v_B^1(q') - v_B^1(q) \leq (q' - q)[u_H - u_L]$ for all $q' > q$; (*ii*) $\delta v_B^1(Q_+^2(q_0)) \leq u_L - p_\ell$; and (*iii*) $(1-q)F(p_\ell/v_L^1(Q_+^2(q)))$ is decreasing in q. The second inequality follows from (*iii*) and the definition of Q_+^2. Condition (35) is the analogue of (32) to 2-step equilibria. Now let $\eta^2(q,\delta) = \pi_h^B(q) - \pi_\ell^B(q, \delta, v_L^1(Q_+^2(q)), v_B^1(Q_+^2(q)))$. It follows immediately from (35) that $\eta^2(q,\delta)$ is strictly increasing in q. The same argument as in the proof of Lemma 10 then shows that the desired result holds.

Now, we establish the properties of ξ^2. Since $\eta^2(q,\delta)$ is strictly increasing in q and strictly decreasing in δ, an argument similar to the one used in the proof of Lemma 7 shows that for each $q_0 \in [\underline{q}^2, \bar{q}^2) \cap (0,1)$, there exists a unique $\delta^2 = \delta^2(q_0) \in [\underline{\delta}, \bar{\delta})$, which is continuous and increasing in q_0, such that $\eta^2(q_0, \delta) \geq 0$ if, and only if $\delta \leq \delta^2(q_0)$. Thus, $\xi^2(q_0) = F(\delta^2(q_0))$ is continuous and increasing in q_0. Note that $\lim_{q_0 \uparrow \bar{q}^2} \xi^2(\bar{q}^2) = \xi^1(\bar{q}^2)$, since $v_L^1(\bar{q}^1) = v_L^0$, $\bar{q}^1 = q^+(\bar{q}^2, v_L^0)$, and $\lim_{q_0 \uparrow \bar{q}^1} v_B^1(q_0) = v_B^0(\bar{q}^1)$ imply that $\lim_{q_0 \uparrow \bar{q}^2} \eta^2(q_0, \delta) = \eta^1(\bar{q}^2, \delta)$.

For the properties of v_L^2, first note that Q_+^2 and v_L^1 continuous in q_0 imply that v_L^2 is also continuous in q_0. Moreover,

$$\lim_{q_0 \uparrow \bar{q}^2} v_L^2(q_0) = \xi^1\big(\bar{q}^2\big)p_h + \big(1 - \xi^1\big(\bar{q}^2\big)\big)\int_{\underline{\delta}}^{\bar{\delta}} \max\big\{p_\ell, \delta v_L^0\big\}dF(\delta) = v_L^1\big(\bar{q}^1\big).$$

To finish, note that $v_L^2(q_0) \le v_L^1(Q_+^2(q_0))$ follows from the fact that $v_L^1(q) \le v_L^0(q^+(q, v_L^0)) = v_L^0$ and that $\xi^2(q_0) \le \xi^1(Q_+^2(q_0))$ by (34) in the proof of Lemma 10. □

Proof of Proposition 2

We proceed by induction. Suppose there exist $k \ge 3$ and sequences $\{\underline{q}^s\}_{s=0}^{k-1}$ and $\{\bar{q}^s\}_{s=0}^{k-1}$ such that:

(A1) $\bar{q}^0 = 1$ and $\underline{q}^s \le \underline{q}^{s-1} < \bar{q}^s < \bar{q}^{s-1}$ for all $s \in \{1, \dots, k-1\}$;
(A2) An s-step equilibrium, with $s \in \{0, \dots, k-1\}$, exists if, and only if, $q_0 \in [\underline{q}^s, \bar{q}^{s-1}) \cap (0, 1)$.

Moreover, suppose that for each $s \in \{0, \dots, k-1\}$, there exist functions $v_B^s(q_0)$ and $v_L^s(q_0)$, and a map $Q_+^s(q_0)$, such that:

(A3) $Q_+^s(q_0)$ is the value of q_1 in an s-step equilibrium if the initial fraction of type H sellers is q_0;
(A4) Given $q_0 \in [\underline{q}^s, \bar{q}^s) \cap (0, 1)$, the payoffs to buyers and type L sellers in an s-step equilibrium are $v_B^s(q_0)$ and $v_L^s(q_0)$, respectively;
(A5) For all $s \in \{2, \dots, k-1\}$, if $q' = Q_+^s(q_0)$, then

$$\begin{aligned}\eta^{s-1}(q', \delta) &= \pi_h^B(q') - \pi_\ell^B\big(q', \delta, v_L^{s-2}\big(Q_+^{s-1}(q')\big), v_B^{s-2}\big(Q_+^{s-1}(q')\big)\big)\\ &\ge \eta^s(q_0, \delta) = \pi_h^B(q_0) - \pi_\ell^B\big(q_0, \delta, v_L^{s-1}(q'), v_B^{s-1}(q')\big)\end{aligned}$$

for all $q_0 \in [\underline{q}^s, \bar{q}^s) \cap (0, 1)$ and $\delta \in [\underline{\delta}, \bar{\delta}]$;
(A6) v_B^s is continuous in q_0, with $v_B^s(q_0') - v_B^s(q_0) \le (q_0' - q_0)[u_H - u_L]$ for all $q_0' > q_0$;
(A7) v_L^s is continuous and increasing in q_0, $v_L^s(\bar{q}^s) = \lim_{q_0 \to \bar{q}^s} v_L^s(q_0) = v_L^{s-1}(\bar{q}^s)$ and $v_L^s(q_0) \le v_L^{s-1}(Q_+^s(q_0))$ for all q_0.

Finally, suppose that:

(A8) For each $s \in \{1, \dots, k-1\}$, $\underline{q}^s = 0$ if, and only if, $p_\ell / v_L^{s-1}(\underline{q}^{s-1}) \ge \bar{\delta}$.

Conditions (A1) to (A8) are true when $k = 3$ by Lemmas 6, 7, 8, 10 and 11; condition (A5) reduces to (34) in the proof of Lemma 8 when $s = 2$. In what follows, we show that $p_\ell / v_L^{k-1}(\bar{q}^{k-1}) < \bar{\delta}$ implies that there exist cutoffs $\underline{q}^k$ and $\bar{q}^k$, payoff functions $v_B^k(q_0)$ and $v_L^k(q_0)$, and a map $Q_+^k(q_0)$ such that (A1) to (A8) are also satisfied when $s = k$.

From the discussion of 2-step equilibria in the main text, it is easy to see that (22) together with the following two conditions are necessary and sufficient for a k-step equilibrium to exist:

$$q' \in \big[\underline{q}^{k-1}, \bar{q}^{k-1}\big) \cap (0, 1); \tag{36}$$

$$\pi_h^B(q_0) < \pi_\ell^B\big(q_0, \bar{\delta}, v_L^{k-1}(q'), v_B^{k-1}(q')\big). \tag{37}$$

We claim that (22) and (36) imply (37). Indeed, suppose (22) and (36) hold and let $\eta^k(q_0, \delta) = \pi_h^B(q_0) - \pi_\ell^B(q_0, \delta, v_L^{k-1}(q'), v_B^{k-1}(q'))$. First note that (31) in the proof of Lemma 10 implies that

$$\begin{aligned}&\pi_\ell^B\big(q_0, \delta, v_L^{k-1}(q'), v_B^{k-1}(q')\big)\\ &\quad= \delta \pi_h^B(q_0) + (1 - q_0) F\left(\frac{p_\ell}{v_L^{k-1}(q')}\right)\big[u_L - p_\ell - \delta(u_L - p_h)\big]\end{aligned}$$

$$+\delta\left\{q_0+(1-q_0)\left[1-F\left(\frac{p_\ell}{v_L^{k-1}(q')}\right)\right]\right\}\left[v_B^{k-1}(q')-\pi_h^B(q')\right].$$

Similarly, one can show that if $q''=Q_+^{k-1}(q')$, then

$$\begin{aligned}&\pi_\ell^B\left(q',\delta,v_L^{k-2}(q''),v_B^{k-2}(q'')\right)\\&\quad=\delta\pi_h^B(q')+(1-q')F\left(\frac{p_\ell}{v_L^{k-2}(q'')}\right)\left[u_L-p_\ell-\delta(u_L-p_h)\right]\\&\qquad+\delta\left\{q'+(1-q')\left[1-F\left(\frac{p_\ell}{v_L^{k-2}(q'')}\right)\right]\right\}\left[v_B^{k-2}(q'')-\pi_h^B(q'')\right].\end{aligned}$$

Now observe that if $q'''=Q_+^2(q'')$, then

$$\begin{aligned}v_B^{k-1}(q')-\pi_h^B(q')&=\int_{\underline{\delta}}^{\bar{\delta}}\max\left\{0,\pi_h^B(q')-\pi_\ell^B\left(q',\delta,v_L^{k-2}(q''),v_B^{k-2}(q'')\right)\right\}dF(\delta)\\&\geq\int_{\underline{\delta}}^{\bar{\delta}}\max\left\{0,\pi_h^B(q'')-\pi_\ell^B\left(q'',\delta,v_L^{k-3}(q'''),v_B^{k-2}(q''')\right)\right\}dF(\delta)\\&=v_B^{k-2}(q'')-\pi_h^B(q''),\end{aligned}$$

where the inequality follows from (A5). Therefore,

$$\begin{aligned}&\pi_\ell^B\left(q_0,\delta,v_L^{k-1}(q'),v_B^{k-1}(q')\right)-\pi_\ell^B\left(q',\delta,v_L^{k-2}(q''),v_B^{k-2}(q'')\right)\\&\quad\geq\delta\left[\pi_h^B(q_0)-\pi_h^B(q')\right]+\lambda\left[(1-q_0)F\left(\frac{p_\ell}{v_L^{k-1}(q')}\right)-(1-q')F\left(\frac{p_\ell}{v_L^{k-2}(q'')}\right)\right],\end{aligned}$$

where $\lambda=\{u_L-p_\ell-\delta(u_L-p_h)-\delta[v_B^{k-1}(q'')-\pi_h^B(q'')]\}\geq u_L-p_\ell-\delta v_B^{k-1}(q'')>0$. Given that $v_L^{k-1}(q')<v_L^{k-2}(q'')$ by (A6) and that $q'\geq q_0$, we can then conclude that

$$\pi_\ell^B\left(q',\delta,v_L^{k-2}(q''),v_B^{k-2}(q'')\right)-\pi_\ell^B\left(q_0,\delta,v_L^{k-1}(q'),v_B^{k-1}(q')\right)<\pi_h^B(q')-\pi_h^B(q_0).$$

Consequently, $\eta^{k-1}(q',\delta)>\eta^k(q_0,\delta)$ for all $\delta\in[\underline{\delta},\bar{\delta}]$. In particular, since $\eta^{k-1}(q',\bar{\delta})\leq 0$ for all $q'\in[\underline{q}^{k-1},\bar{q}^{k-1})\cap(0,1)$, we have that $\eta^k(q_0,\bar{\delta})<0$ as well, so that (37) is indeed satisfied.

Suppose now that $p_\ell/v_L^{k-1}(\bar{q}^{k-1})<\bar{\delta}$ and define the cutoffs $\underline{q}^k$ and $\bar{q}^k$ to be such that: (*i*) $q^+(\underline{q}^k,v_L^{k-1}(\underline{q}^{k-1}))=\underline{q}^{k-1}$ if $p_\ell/v_L^{k-1}(\underline{q}^{k-1})<\bar{\delta}$ and $\underline{q}^k=0$ otherwise; and (*ii*) $q^+(\bar{q}^k,v_L^{k-1}(\bar{q}^{k-1}))=\bar{q}^{k-1}$. It is immediate to see $0<\bar{q}^k<\bar{q}^{k-1}$. Since $\underline{q}^{k-1}=0$ if, and only if, $p_\ell/v_L^{k-2}(\underline{q}^{k-2})\geq\bar{\delta}$ (by (A8)) and $v_L^{k-1}(\underline{q}^{k-1})\leq v_L^{k-2}(\underline{q}^{k-2})$ (by (A6)), we have that $\underline{q}^k\leq\underline{q}^{k-1}$. Now note that if $\underline{q}^{k-1}=0$, then (trivially) $\bar{q}^k>\underline{q}^{k-1}$. Suppose then that $\underline{q}^{k-1}>0$. Given that $\bar{q}^{k-1}>\underline{q}^{k-2}$, we have that $v_L^{k-1}(\bar{q}^{k-1})=v_L^{k-2}(\bar{q}^{k-1})\geq v_L^{k-2}(\underline{q}^{k-2})$. Thus,

$$q^+\left(\bar{q}^k,v_L^{k-2}(\underline{q}^{k-2})\right)\geq q^+\left(\bar{q}^k,v_L^{k-1}(\bar{q}^{k-1})\right)=\bar{q}^{k-1}>\underline{q}^{k-2}=q^+\left(\underline{q}^{k-1},v_L^{k-2}(\underline{q}^{k-2})\right),$$

from which we obtain that $\bar{q}^k>\underline{q}^{k-1}$; recall that $q^+(q,v_L)$ is strictly increasing in q when $p_\ell/v_L<\bar{\delta}$. Finally, the same argument used in the proof of Lemma 8, just replace the superscripts "1" and "2" with "$k-1$" and "k," respectively, shows that: (*i*) there exists a k-step equilibrium if, and only if, $q_0 \in [\underline{q}^k, \bar{q}^k) \cap (0,1)$; (*ii*) for each $q_0 \in [\underline{q}^k, \bar{q}^k) \cap (0,1)$, there exists a unique $q' = Q_+^k(q_0) \in [\underline{q}^{k-1}, \bar{q}^{k-1}) \cap (0,1)$ such that q' is the value of q_1 in any k-step equilibrium when the initial fraction of type H sellers is q_0; and (*iii*) the map Q_+^k is continuous and strictly increasing. Thus, (A1), (A2), (A3), (A5), and (A8) are valid for $s = k$.

To finish the induction step, let v_B^k and v_L^k be given by (23) and (24), respectively, where $\xi^k(q_0) = \int_{\underline{\delta}}^{\bar{\delta}} \mathbb{I}\{\eta^k(q_0, \delta) \geq 0\} dF(\delta)$. By construction, for each $q_0 \in [\underline{q}^k, \bar{q}^k) \cap (0,1)$, $v_B^k(q_0)$ and $v_L^k(q_0)$ are, respectively, the payoffs to buyers and type L sellers in a k-step equilibrium (so that (A4) holds when $s = k$), and $\xi^k(q_0)$ is the fraction of buyers who offer p_h in the first period of trade in a k-step equilibrium. The same argument used in the proof of Lemma 11 shows that $\xi^k(q_0)$ is increasing in q_0 and that (A6) and (A7) hold when $s = k$; once again just replace the superscripts "1" and "2" with "$k-1$" and "k," respectively.

The induction process described above continues until k is such that $p_\ell / v_L^k(\bar{q}^k) \geq \bar{\delta}$, if such a k exists. We conclude the proof by showing that such a k indeed exists, so that $K = \max\{k : p_\ell / v_L^{k-1}(\bar{q}^{k-1}) < \bar{\delta}\}$. Suppose not. In this case, there exists a strictly decreasing sequence $\{\bar{q}^k\}_{k=0}^\infty$ such that if $q_0 < \bar{q}^k$, then there exists an s-step equilibrium with $s \geq k$ when the initial fraction of type H sellers in the market is q_0. Since the market clears in a finite number of periods in any equilibrium, it must then be that $\lim_{k\to\infty} \bar{q}^k = 0$. In particular, there exists $k_0 \in \mathbb{N}$ such that $\pi_h^B(\bar{q}^k) < 0$ for all $k \geq k_0$. This implies that $\xi^k(\bar{q}^k) = 0$ for all $k \geq k_0$, as not even a myopic buyer finds it optimal to offer p_h when the expected payoff from doing so is negative. Therefore, $\lim_{k\to\infty} v_L^k(\bar{q}^k) = \lim_{k\to\infty} v_L^{k-1}(\bar{q}^k) = p_\ell$, a contradiction.

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