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Directed search with multi-vacancy firms

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Abstract

I construct a directed search model in which firms decide whether to enter a market and how many positions to create. Within this framework, the number of firms and the size of each firm are determined endogenously, wages play an allocative role in the matching process, and the frictions inherent in this process derive from the equilibrium behavior of workers and firms. I characterize the (unique) equilibrium. Comparative statics generate testable implications for cross-sectional variation in matching efficiency, as well as the dynamic behavior of vacancies and unemployment. Moreover, allowing for ex-ante heterogeneity across firms, the model can easily and naturally generate the observed relationship between firm size, wages, profitability, and hiring.

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1. Introduction

Suppose there are u workers looking for a job and v positions to be filled. A central question in search and matching models of the labor market is: how many matches will be formed, and at what terms of trade? The point of departure for the current paper is the observation that, for ex-

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ample, whether a single firm is posting v vacancies or v firms are posting a single vacancy would seem to be of critical importance for understanding the frictions in the market, and predicting the number of matches formed, wages, and total output. In other words, in order to answer the question posed above, an economist may need to know not only the *number* of vacancies posted but the *distribution* of these vacancies across firms as well. At the very least, the concentration of vacancies will determine the extent to which firms are competing with one another for workers, and thus the wages they offer potential applicants, as well as the application behavior of job seekers, and thus the level of frictions inherent in the matching process.

In this paper, I construct a framework in which firms choose whether or not to enter an industry, how many vacancies to create, and what wages to offer potential employees; workers then observe each firm's decisions and choose an application strategy to maximize expected utility. In this environment, both the number of firms and the size of each firm are determined endogenously, wages play an allocative role in the matching process, and the frictions inherent in this process derive from the equilibrium behavior of workers and firms. I characterize the (unique) equilibrium. Comparative statics generate a rich set of testable implications for cross-sectional variation in matching efficiency, as well as the dynamic behavior of vacancies and unemployment. Moreover, allowing for ex-ante heterogeneity across firms, I show that the model can easily and naturally generate the relationship between firm size, wages, profitability, and hiring that is observed in the data.

My approach belongs to the literature on directed search.[2] In the canonical directed search model, firms with a single vacancy post wages, and then workers observe these wages and apply strategically to a single firm. The key restriction is that workers cannot coordinate their application behavior – i.e. they must play symmetric strategies – so that in equilibrium multiple applicants may arrive at one firm (in which case one worker is chosen at random), while another firm may receive no applicants. Thus, a worker's optimal application strategy balances the trade-off between the wage being offered and the probability of being hired, while a firm's optimal wage-posting strategy balances the trade-off between the wage it must pay and the probability of filling its vacancy.

This model is appealing because the micro-foundations of both the wage-formation and matching processes are derived explicitly, and the model can accommodate the introduction of various types of heterogeneity on both sides of the market. This is especially important in light of the recent arrival of data collected at the worker/firm level, and the variety of new "stylized facts" emerging from this data. However, despite the potential advantages of the directed search paradigm, the assumption that each firm can open at most a single position is quite restrictive, in the sense that it offers no distinction between a firm and a vacancy, and thus little scope for the analysis of how firm size, firm growth, and market structure affect matching, wages, and output.[3] In Section 2, as a first step, I propose and solve a directed search model in which firms choose how many vacancies to create; for tractability, I restrict the choice set to one or two vacancies.

[2] The literature on directed search can be traced back to Peters [22,23] and Montgomery [19], and was further developed by Shimer [28], Moen [18], and Burdett, Shi, and Wright [5].

[3] Indeed, the first models in this literature made a variety of severe assumptions, many of which have been relaxed in recent research. For example, Albrecht, Gautier, and Vroman [3] and Galenianos and Kircher [9] relax the common assumption that workers can only apply to a single firm. Julien, Kennes and King [13], Coles and Eeckhout [7], and Shi [27] relax the assumption that firms post and commit to wages by considering alternative mechanisms. Acemoglu and Shimer [2] allow for risk-aversion and ex-ante capital investment, while Shi [25] allows for heterogeneous skills among workers. Guerrieri, Shimer, and Wright [10], Menzio [16], and Lester [15] introduce informational frictions.

Holding constant the number of firms, I fully characterize the distribution of vacancies, wages, and the resulting number of matches that arise in equilibrium. A key result is that the endogenously generated matching function becomes more efficient at matching workers with firms as the distribution of vacancies becomes more concentrated.

In Section 3, I introduce entry by firms, so that both the number and the size-distribution of firms are endogenous. I allow for uncertainty over aggregate productivity, which I assume is realized after the firms' entry decision but before they choose how many vacancies to post. This modeling specification is meant to capture (within the context of a static model) the notion that a firm's entry decision is, in general, based on expected market conditions while a firm's decision to expand or contract is typically in response to realized market conditions. After characterizing the unique equilibrium, comparative statics uncover a number of interesting results, providing a systematic relationship between features of the economic environment and the market's ability to efficiently match workers and firms.

In the cross section, the model suggests that more efficient matching should be associated with environments that either discourage entry or encourage existing firms to expand, such as high entry costs, low costs of recruiting additional workers, or high realized values (relative to expected values) of labor productivity. Over time, the model suggests that *how* vacancies are created has important implications for equilibrium outcomes; the efficiency of the matching technology depends on the extent to which new positions are created along the extensive margin (by the entry of new firms) or the intensive margin (by existing firms posting multiple vacancies). When vacancy creation occurs along the intensive margin, the number of matches formed increases for *two* reasons: because there are more vacancies per worker, as usual, and because the matching function becomes more efficient. Thus, incorporating both margins into the canonical model of directed search amplifies the response of the predicted number of matches to changes in market tightness (the ratio of vacancies to unemployed workers). More generally, the model identifies a relationship between the *type* of shock that causes an expansion and the nature of the expansion itself. Shocks that tend to cause entry by many small (single-vacancy) firms will be associated with less substantial gains in employment, while those that cause vacancy creation by fewer large (two-vacancy) firms will lead to more substantial gains.

These theoretical relationships appear roughly consistent with a variety of stylized facts emerging from the data. In the cross-section, vacancy-level data indicates that matching is indeed more efficient in labor markets with a higher concentration of vacancies, and at firms who are expanding rapidly (presumably by posting multiple vacancies). Over time, recent studies suggest that (i) small firms create relatively more jobs in recessions, while large firms create relatively more jobs in expansions; and (ii) the efficiency of the matching function is pro-cyclical. The theory presented here could provide micro-foundations for these as-yet-unidentified shocks to the efficiency of the matching function. These insights, and more, are discussed in Section 3.3.

Finally, in Section 4, I extend the model to allow for firms to be ex-ante heterogeneous with respect to their level of productivity, and show that this framework can easily and naturally generate the positive relationship between firm size, wages, profitability, and hiring rates that has been observed in the data. The mechanism is straightforward: since productive firms generate high output per worker and have a high opportunity cost of not hiring, they have incentive to both open multiple vacancies and post high wages associated with those vacancies. Since workers apply to high wage firms with greater probability, *and* since matching is more efficient at multiple vacancy firms, these firms hire with high probability and, in equilibrium, are more profitable than less productive firms.

1.1. Related models

This paper is closely related to Burdett, Shi, and Wright [5], who first point out that the firm-size distribution could affect matching frictions and resulting levels of wages and unemployment. However, they do so by allowing for exogenous heterogeneity in the number of vacancies posted by each firm. Therefore, though they identify this type of heterogeneity as being potentially important in determining equilibrium outcomes, they are silent on the *sources* of firm-size heterogeneity and thus on the *causes* of changes in equilibrium wages, matching, and output. Shi [26] considers a directed search environment in which firm size is determined endogenously, but he assumes that firms can only post a single vacancy per period. Thus, in his model the distribution of vacancies in each period remains degenerate, and the firm-size distribution has no effect on the efficiency of the matching technology.[4] The current paper builds on these models by endogenizing the vacancy distribution across firms, introducing an entry decision, and allowing for ex-ante heterogeneity in order to study the behavior of the matching function and wages, both in the cross-section and over time.

Hawkins [11] also considers a competitive search environment (more in the spirit of Shimer [28] and Moen [18]) with multi-worker firms. Several assumptions distinguish the two papers. First, I assume that the marginal cost of creating each additional position is positive and is incurred by the firm before hiring any workers, while he assumes that firms pay a fixed entry cost and then can create any number of positions ex-post at zero marginal cost. Of course, which of these two assumptions is more appropriate depends largely on the particular industry or occupation. My framework would be most natural in situations where a firm must incur either capital expenditures (e.g. acquire additional office space) or labor expenditures (e.g. expand human resources) in order to create each new position. Second, while I focus on the endogenously generated matching function and its sensitivity to the economic environment, Hawkins treats the matching function as exogenous and focuses his efforts on the question of efficiency.

2. Benchmark model

There is a fixed measure of unemployed workers and a fixed measure of firms, both of which are ex-ante homogeneous. Let r denote the ratio of unemployed workers to firms. The game proceeds in two stages. In stage one, firms face two decisions. The first decision is how many vacancies (or positions) to create. For simplicity, I assume that they can either create one vacancy at cost C_1 or two vacancies at cost $C_1 + C_2$. A firm produces output y_1 if matched with one worker, and $y_1 + y_2$ if matched with two workers. If the cost function is concave ($0 \leqslant C_2 \leqslant C_1$) and the production function is convex ($0 \leqslant y_1 \leqslant y_2$), then in equilibrium all firms will either post two vacancies or remain inactive (i.e. post no vacancies). The more interesting case, in which some firms post a single vacancy and others post two vacancies, thus requires either convexity in the costs of posting vacancies or concavity in the production function. As it is analytically more convenient, I will choose the former, and assume that $y_1 = y_2 \equiv y > 0$, $C_1 = 0$, and $0 < C_2 < y$.[5]

[4] In the same model, Shi [26] allows for firms to hold multiple units of a good to sell in a product market, so that they compete in both price and capacity. This is similar in spirit to the current exercise.

[5] All of the results presented below remain true under the alternative assumption of decreasing returns in the production technology and constant returns in the cost of posting vacancies.

The second decision that firms face is the wage at which they are committed to paying their workers. I assume that a firm with two vacancies sets the same wage for both positions.[6]

In stage two, each worker observes the wage and the number of vacancies (hereafter referred to as the *capacity*) at every firm and applies to the firm that offers the highest expected payoff. Note that I have assumed workers apply to firms rather than specific positions. This is important: if workers applied to specific positions instead, then a single firm posting two positions would be identical to two firms posting a single position, and the distinction between a firm and a position would essentially be eliminated.[7] I also assume that workers can only apply to a single firm. If the number of workers that arrive at a particular firm exceeds capacity, the firm allocates the position(s) at random, with each worker receiving a job with equal probability. Therefore, the expected payoff to the worker from applying to each firm depends on both the posted wage and the probability of receiving the job.

The model is solved backwards. First, I characterize the symmetric strategy Nash equilibrium in the second stage sub-game, taking as given the distribution of wages and capacities.[8] In this equilibrium, the workers' strategies are such that they are indifferent between every firm which they apply to with strictly positive probability: the expected number of workers at each firm, and thus the probability of a match at each firm, adjusts to exactly offset any difference in wages. Therefore, given the behavior of other firms, the equilibrium of the second stage sub-game provides a precise relationship between an individual firm's decision (their capacity and wage) and the probability that they will hire a worker. This allows firms to easily calculate their expected payoffs, and thus the second step – characterizing equilibrium at the first stage of the game – is standard.

2.1. Stage two: optimal job search

In order to characterize the optimal behavior of workers in the second stage sub-game, it is first necessary to derive the expected value of a worker visiting a firm with capacity $k \in \{1, 2\}$ and wage w. To do so, it is convenient to consider the case of a finite number of workers and firms in fixed proportion, and then allow the number of agents to tend to infinity (this follows

[6] If firms could post different wages for different positions, it turns out that the strategy of posting one wage for both positions would still be an equilibrium, though there would be many others. Importantly, in all of these equilibria, the results below concerning matching, efficiency, and output remain unchanged (even if $y_1 \neq y_2$). I illustrate this point, as well as the claim in footnote 5, in Appendix A.

[7] There are several justifications for assuming that workers apply to firms rather than specific positions. First, this model can be interpreted as a labor market for a particular type of worker, so that positions are homogeneous. For example, a restaurant chain that has opened a new branch will often post vacancies for waiters or waitresses; an applicant simply applies to the restaurant. Moreover, even if I were to introduce some heterogeneity (across workers and positions), in general firms do have some ability to coordinate hiring decisions; an Economics department that posts an opening for a macroeconomist and an opening for an econometrician may be able to hire two macroeconomists if they can't find a suitable econometrician. One could imagine a simple extension of the current model in which a firm with two positions that received two or more applicants for one position and zero for the other could hire a second worker with some probability x. The framework developed here would then be the special case of $x = 1$, though the basic mechanism driving the results would be preserved for any $x > 0$.

[8] Restricting attention to symmetric strategies for workers is standard in this literature, and crucial for generating a coordination friction. This assumption is generally justified by assuming, as I do, that the labor market is large and workers are anonymous, thus making it difficult to coordinate on asymmetric strategies. See both Burdett, Shi, and Wright [5] and Shimer [29] for a more detailed discussion.

the analysis in Burdett, Shi, and Wright [5]). To that end, let u and f denote the number of unemployed workers and firms, respectively, with $r = u/f$.

Consider the problem of worker i who is considering applying to a firm j with a single vacancy posted. If the $u-1$ other workers apply to this firm with probability θ^j, then the probability that worker i is hired is

$$\Lambda_1(\theta^j) = \sum_{n=0}^{u-1}\left[\frac{(u-1)!}{n!(u-1-n)!}\right](\theta^j)^n(1-\theta^j)^{u-1-n}\frac{1}{n+1},$$

which simplifies to

$$\Lambda_1(\theta^j) = \left[1-(1-\theta^j)^u\right]/(u\theta^j). \tag{1}$$

Similarly, if firm j has two vacancies, the probability that worker i is hired is

$$\Lambda_2(\theta^j) = (1-\theta^j)^{u-1} + \sum_{n=1}^{u-1}\left\{\left[\frac{(u-1)!}{n!(u-1-n)!}\right](\theta^j)^n(1-\theta^j)^{u-1-n}\frac{2}{n+1}\right\}.$$

This can be simplified to

$$\Lambda_2(\theta^j) = \frac{2}{u\theta^j}\left[1-(1-\theta^j)^u\right] - (1-\theta^j)^{u-1}. \tag{2}$$

Of course, the expected payoff to each worker from applying to a firm with capacity k^j and wage w^j when all other workers are applying with probability θ^j is equal to $\Lambda_{k^j}(\theta^j)w^j$. Therefore, a symmetric strategy Nash equilibrium of the stage two game with a finite number of players is a strategy $\theta^* \equiv (\theta^{1*}, \ldots, \theta^{f*}) \in [0,1]^f$ with the properties that $\sum_{j=1}^f \theta^{j*} = 1$ and, for all j, $\theta^{j*} > 0$ only if $\Lambda_{k^j}(\theta^{j*})w^j \geqslant \Lambda_{k^{j'}}(\theta^{j'*})w^{j'}$ $\forall j' \neq j$.

Now consider the stage two sub-game as the number of players tends to infinity. Let $q^j = \theta^j u$ denote the expected number of applicants, or *queue length*, at firm j. Using (1) and (2), it is a standard calculation to establish that as the number of agents tends to infinity, the expected payoffs to a worker from applying to a firm j that has posted $k^j \in \{1,2\}$ vacancies at wage w^j, and has an expected queue length q^j, is

$$U_1(w^j, q^j) = \left[\frac{1-e^{-q^j}}{q^j}\right]w^j, \tag{3}$$

$$U_2(w^j, q^j) = \left[\frac{2}{q^j}\left(1-e^{-q^j}\right) - e^{-q^j}\right]w^j. \tag{4}$$

An individual worker will apply to firm j only if

$$U_{k^j}(w^j, q^j) = \max_{j'}\left\{U_{k^{j'}}(w^{j'}, q^{j'})\right\} \equiv U, \tag{5}$$

where the variable U is often referred to as the *market utility*. Let $Q_{k^j}(w^j, U)$ equal the unique value of q^j satisfying (5) for $w^j \geqslant U > 0$, and zero for $w^j < U$.[9]

[9] This definition of queue lengths is standard in this literature. However, it is worth noting an implicit assumption of anonymity: workers apply with equal probability to all firms with the same capacity and the same wage. This stands in contrast to the game with a finite number of agents, which allowed for workers to condition their application strategy on the name j of a particular firm. Also, notice that restricting attention to $U > 0$ is without loss, as this will be true for all equilibrium with $r < \infty$.

Definition 1. Given a distribution $H(w^j, k^j)$ of wages and capacities across firms, a symmetric equilibrium of the second stage sub-game is a market utility U^* and expected queue lengths q^{j*} such that (i) $q^{j*} = Q_{k^j}(w^j, U^*)$ for all j, and (ii) $\int_j Q_{k^j}(w^j, U^*)\, dH(w^j, k^j) = r$.

Therefore, given any distribution of wages and capacity, equilibrium at the second stage assigns expected queue lengths such that workers are indifferent between applying to every firm j such that $q^{j*} > 0$.

2.2. Stage one: profit maximization

Now consider the problem facing a representative firm at the first stage. Maintain the assumption that there is a measure of workers and firms denoted by u and f, respectively, with $r = u/f$. In stage one, a firm takes the market utility as exogenous and understands that, for any capacity choice $k \in \{1, 2\}$ and wage w_k, the expected number of applications will be $Q_k(w_k, U)$. Therefore, a firm with one vacancy will set a wage w_1 so as to solve the following profit-maximization problem:

$$\pi_1^*(U) = \Big\{ \max_{w_1} \pi_1(w_1, U) = \big[1 - e^{-Q_1(w_1,U)}\big](y - w_1) \Big\}. \tag{6}$$

The profits of a single vacancy firm are the product of the probability that the firm receives at least one applicant, $1 - e^{-Q_1(w_1,U)}$, and the surplus to the firm, $y - w_1$.[10] Similarly, a firm with two vacancies sets a wage w_2 to solve the profit-maximization problem:

$$\begin{aligned}\pi_2^*(U) = \Big\{ \max_{w_2} \pi_2(w_2, U) = \big\{2\big[1 - e^{-Q_2(w_2,U)}\big] \\ - Q_2(w_2, U) e^{-Q_2(w_2,U)}\big\}(y - w_2) - C_2 \Big\}. \end{aligned} \tag{7}$$

Definition 2. Let $\pi^*(U^*) = \max\{\pi_1^*(U^*), \pi_2^*(U^*)\}$. An equilibrium at stage one is a distribution $H^*(w^j, k^j)$ of wages and capacities across firms, a market utility U^*, and queue lengths q^{j*} such that (i) $\pi_{k^j}(w^j, U^*) = \pi^*(U^*)$ for all (w^j, k^j) such that $dH^*(w^j, k^j) > 0$; (ii) $\pi_{k^j}(w^j, U^*) \leqslant \pi^*(U^*)$ for all (w^j, k^j) such that $dH^*(w^j, k^j) = 0$; and (iii) U^* and q^{j*} constitute a symmetric equilibrium of the second stage sub-game given $H^*(w^j, k^j)$.

2.3. Characterization of equilibrium

To characterize the equilibrium, consider first the profit maximization problem of a firm with capacity $k = 1$. Equilibrium in the second stage requires $U_1(w_1, q_1) = U$, which implies a one-to-one relationship between the wage a firm posts, w_1, and the expected queue length q_1. Solving (3) for w_1 and substituting into (6), one can re-write the problem of a firm with $k = 1$ in terms of the equilibrium queue length in the second stage game:[11]

$$\max_{q_1} y\big(1 - e^{-q_1}\big) - q_1 U. \tag{8}$$

[10] The probability that a seller with $k = 1$ receives at least one application is the product of the expected number of applicants, q, and the probability that each applicant is hired, $(1 - e^{-q})/q$. Similar reasoning can be used to understand (7).

[11] This formulation ignores the possibility that the firm sets a wage such that no workers apply (i.e. $q_1 = 0$), which cannot be consistent with equilibrium behavior for any market utility $U < y$.

Note that the firm's problem is strictly concave in q_1. Therefore, if a firm opens up a single vacancy, there exists a unique queue length satisfying the first-order condition

$$U = ye^{-q_1}, \tag{9}$$

and thus a unique profit-maximizing wage

$$w_1 = \frac{q_1 ye^{-q_1}}{1 - e^{-q_1}}. \tag{10}$$

Similarly, one can use (4) and (7) to re-write the problem of a firm with two vacancies:

$$\max_{q_2} \bigl[2\bigl(1 - e^{-q_2}\bigr) - q_2 e^{-q_2}\bigr] y - q_2 U - C_2. \tag{11}$$

Again, there is a unique profit-maximizing value of q_2 satisfying the first-order condition

$$U = ye^{-q_2}(1 + q_2), \tag{12}$$

and this uniquely determines the optimal wage

$$w_2 = \frac{q_2(1 + q_2) ye^{-q_2}}{2(1 - e^{-q_2}) - q_2 e^{-q_2}}. \tag{13}$$

To summarize, conditional on k, there exists a unique profit-maximizing wage w_k and queue length q_k. Hence the relevant strategy of a firm can be simplified to a single variable, ϕ, which I define as the probability that a firm opens a single vacancy. Naturally, then, with probability $1 - \phi$ a firm opens two vacancies.[12]

Since there will only be two different types of firms, a worker's strategy can also be reduced to a single decision: whether to apply to a firm with one vacancy (a *type* 1 firm) or a firm with two vacancies (a *type* 2 firm). I assume that firms are anonymous, so that a worker applies to each identical firm (i.e. each firm with the same number of vacancies and the same wage) with equal probability. Let $\sigma \in [0, 1]$ denote the probability that a worker applies to a type 1 firm. Feasibility requires that $\phi = 1 \Rightarrow \sigma = 1$ and $\phi = 0 \Rightarrow \sigma = 0$, since workers cannot apply to firms that do not exist. Alternatively, in any equilibrium with $\phi \in (0, 1)$, it must be that $\sigma \in (0, 1)$, and the expected number of workers to apply to type 1 and type 2 firms must satisfy

$$q_1 = (r\sigma)/\phi, \tag{14}$$

$$q_2 = \bigl[r(1 - \sigma)\bigr]/(1 - \phi). \tag{15}$$

Moreover, in any such equilibrium, workers must be indifferent between applying to a type 1 and type 2 firm. Given (9) and (12), $U_1(w_1, q_1) = U_2(w_2, q_2)$ reduces to

$$e^{-q_1} - e^{-q_2}(1 + q_2) = 0. \tag{16}$$

Plugging in (14) and (15), Eq. (16) is an implicit function $\Sigma : \phi \to \sigma$. This function reflects the equilibrium strategy of workers in the second stage sub-game in response to a fraction ϕ of firms posting a single vacancy in the first stage, conditional on the firms posting the optimal wages.

An equilibrium with $\phi \in (0, 1)$ also requires that firms are indifferent between posting one or two vacancies. Using the equilibrium conditions above, $\pi_1^*(U) = \pi_2^*(U)$ reduces to

$$e^{-q_1}(1 + q_1) - e^{-q_2}\bigl[2(1 + q_2) + q_2^2\bigr] + 1 - \frac{C_2}{y} = 0. \tag{17}$$

[12] Note that, since $C_1 = 0$, a firm will never want to post zero vacancies. I consider the entry decision later in the paper.

Eqs. (16) and (17) constitute necessary and sufficient conditions on the pair (q_1, q_2) for an interior equilibrium. The values of q_1 and q_2 that satisfy these conditions then uniquely determine equilibrium strategies (σ, ϕ) through (14) and (15), wages through (10) and (13), and the market utility $U = U_1(w_1, q_1) = U_2(w_2, q_2)$. The following lemma establishes that there is a unique candidate for an interior equilibrium. All proofs can be found in Appendix A.

Lemma 1. *There exists a unique pair* (q_1, q_2) *satisfying* (16) *and* (17): q_2 *is the unique solution to*

$$e^{-q_2}\left[1 + (1 + q_2)\log(1 + q_2)\right] = 1 - C_2/y, \tag{18}$$

and

$$q_1 = q_2 - \log(1 + q_2). \tag{19}$$

For convenience, let $c \equiv C_2/y$ denote the cost of posting a second vacancy relative to output. Also, let $\bar{q}_2(c)$ denote the candidate equilibrium value of q_2 that satisfies (18), and let $\bar{q}_1(c) = \bar{q}_2(c) - \log[1 + \bar{q}_2(c)]$ denote the candidate equilibrium value of q_1.

Proposition 1. *For any* $0 < r < \infty$ *and* $C_2/y \equiv c \in (0, 1)$*, there exists a unique equilibrium. If* $r \geqslant \bar{q}_2(c)$*, then* $q_2^* = r$ *and* $\sigma^* = \phi^* = 0$*. If* $r \leqslant \bar{q}_1(c)$*, then* $q_1^* = r$ *and* $\sigma^* = \phi^* = 1$*. If* $\bar{q}_1(c) < r < \bar{q}_2(c)$*, then* $q_1^* = \bar{q}_1(c)$*,* $q_2^* = \bar{q}_2(c)$*,*

$$\sigma^* = \frac{\bar{q}_1(c)[\bar{q}_2(c) - r]}{r[\bar{q}_2(c) - \bar{q}_1(c)]}, \quad \textit{and} \tag{20}$$

$$\phi^* = \frac{\bar{q}_2(c) - r}{\bar{q}_2(c) - \bar{q}_1(c)}. \tag{21}$$

In this equilibrium, $\partial\phi^*/\partial c > 0$*,* $\partial\sigma^*/\partial c > 0$*,* $\partial\phi^*/\partial r < 0$*, and* $\partial\sigma^*/\partial r < 0$*.*

The conditions on r, C_2, and y that determine ϕ^* and σ^* are intuitive (for the reasoning that follows, it is helpful to note that both $\bar{q}_1(c)$ and $\bar{q}_2(c)$ are increasing functions of c). If the ratio of unemployed workers to firms is sufficiently large, the cost of posting a second vacancy is sufficiently small, or the output from a match is sufficiently high, so that $r \geqslant \bar{q}_2(c)$, the unique equilibrium is $(\sigma^*, \phi^*) = (0, 0)$; all firms post two vacancies and workers apply to these types of firms with probability one. The wage is given by (13), with $q_2 = r$. In this region of the parameter space, the probability of receiving at least two applications is sufficiently high to justify the relatively low costs of posting a vacancy, independent of the number of other type 2 firms. The opposite is true for $r \leqslant \bar{q}_1(c)$: in this region of the parameter space all firms post a single vacancy and the wage is given by (10), with $q_1 = r$. Here the probability of finding a second worker is not high enough to justify the relative cost of the vacancy, independent of the strategies of other firms.

When $\bar{q}_1(c) < r < \bar{q}_2(c)$, the unique equilibrium is a mixed strategy equilibrium; some firms post one vacancy, some firms post two vacancies, and workers apply to each type with strictly positive probability. Wages at type 1 and type 2 firms are given by (10) and (13), respectively, with $q_1 = \bar{q}_1(c)$ and $q_2 = \bar{q}_2(c)$. Indeed, one can show that in equilibrium w_1 is greater than w_2; type 2 firms offer workers a greater chance of being hired, and hence extract a larger portion of the surplus in each match. This prediction is counter-factual: it is well documented that, ceteris paribus, larger firms pay higher wages (Brown and Medoff [4]). I attempt to reconcile the model with this fact in Section 4.

2.4. Matching

In this subsection, I analyze the matching technology that arises in equilibrium. Let $m_k(q_k)$ denote the expected number of matches at a firm with capacity k when the expected queue length is q_k. It was established earlier that

$$m_1(q_1) = 1 - e^{-q_1},$$
$$m_2(q_2) = 2\left(1 - e^{-q_2}\right) - q_2 e^{-q_2}.$$

Therefore, the matching technology can be summarized by

$$M(u, f, \sigma, \phi) = \begin{cases} f[m_1(\frac{u}{f})] & \text{if } \phi = 1, \\ f\{\phi m_1[\frac{u\sigma}{f\phi}] + (1-\phi) m_2[\frac{u(1-\sigma)}{f(1-\phi)}]\} & \text{if } \phi \in (0,1), \\ f[m_2(\frac{u}{f})] & \text{if } \phi = 0 \end{cases}$$

where M denotes the aggregate number of matches formed when a measure f of firms post a single vacancy with probability ϕ, and a measure u of unemployed workers apply to a single-vacancy firm with probability σ. Given the relationship between the number of firms and vacancies,

$$v = f\left[\phi + 2(1-\phi)\right], \tag{22}$$

let

$$\hat{M}(u, v, \phi) = M\left[u, v/(2-\phi), \Sigma(\phi), \phi\right].$$

$\hat{M}$ is the aggregate matching function given equilibrium behavior in the second stage.

Result 1. *For any $u > 0$, $v > 0$, and $\phi \in (0,1)$, $\frac{\partial \hat{M}(u,v,\phi)}{\partial \phi} < 0$.*

Result 1 establishes that the matching function becomes more efficient as the distribution of vacancies becomes more concentrated at fewer firms.[13] The reason is that $2m_1[u/(2f)] < m_2(u/f)$; given the same number of unemployed workers, more matches will be formed if f firms are posting two vacancies than if $2f$ firms are posting a single vacancy. Recall that the matching friction here arises from the assumption that workers cannot coordinate their application strategies, and thus with strictly positive probability there are more workers than vacancies at some locations and more vacancies than workers at others. As vacancies become more concentrated, the frictions associated with coordination failure are relaxed. In the limit, when a single firm posts all vacancies, these frictions vanish entirely: the short side of the market is matched and matching is efficient.

This result suggests that standard reduced-form matching functions that admit only u and v as arguments may be ignoring an important margin.[14] In the next section, I introduce an entry decision by firms, so that both the intensive and extensive margins of vacancy creation are determined endogenously. With both margins, the model provides a link between fundamental

[13] This result formalizes the discussion by Burdett, Shi, and Wright [5], who use a numerical example to identify the potential effects of the distribution of vacancies on matching efficiency.

[14] The standard reduced-form matching function is used in a wide class of models; a standard reference is Mortensen and Pissarides [20].

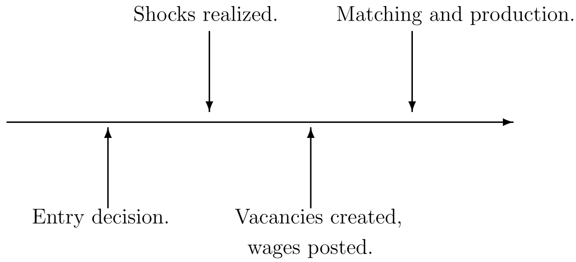

Fig. 1. Timing with entry and uncertainty.

features of the economic environment, the number and distribution of vacancies across firms, and observed results such as the level of employment.

3. Free entry

I now consider entry by firms. I allow for uncertainty over the aggregate productivity parameter y, and assume that the realization of y occurs after the firms' entry decision, but before the firms' capacity choice. This modeling specification is meant to capture, within the context of a static model, the distinction between *entry* and *expansion*. In general, entering an industry and creating a new firm requires a considerable amount of time and expenditure, and this decision is typically based on the profitability of the endeavor over a relatively long time horizon. Thus a firm's entry decision is often based on *expected* market conditions. On the other hand, the amount of time and expenditure required for an existing firm to adjust its labor force is often comparatively small, and such decisions can be adjusted at a relatively greater frequency. Thus a firm's decision to expand or contract is typically in response to *realized* market conditions.

3.1. Extending the model

Suppose now that there is an arbitrarily large mass of firms that can potentially enter the market, and maintain the assumption that there is a mass u of unemployed workers. At the beginning of the game, firms choose whether or not to incur an entry cost $C_e > 0$ and enter the market. After entry, they can post a single vacancy at cost $C_1 = 0$ or two vacancies at cost $C_2 > 0$. Denote the mass of firms that enter by f, and again let $r = u/f$.

After entry has occurred, but prior to the firms' capacity decision, the value of aggregate productivity y is drawn from a distribution $G(y)$ with support $[y_L, y_H]$. After the value of y is realized, the game proceeds as in Section 2: firms choose capacity and post wages, workers search, matches are formed, and production ensues. The timeline in Fig. 1 summarizes.

3.2. Equilibrium with free entry

Once entry has occurred and y is realized, the outcome is described by the equilibrium in which r is taken as exogenous. To characterize equilibrium with endogenous entry, then, one simply needs to identify the value (or, potentially, values) of r such that a firms's expected profit from entering the market is equated to the cost of entry. Let $\Pi(r, y)$ denote the equilibrium profits

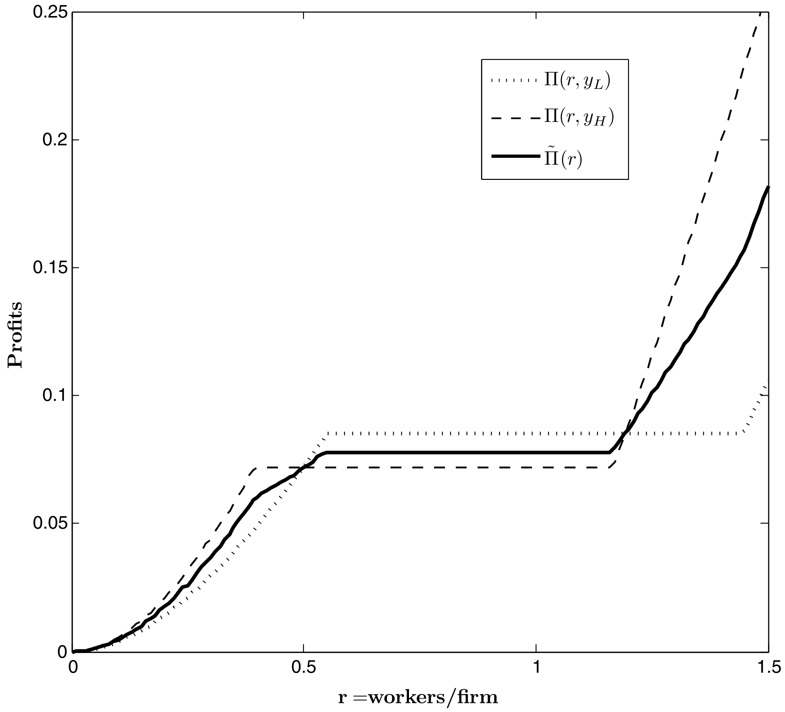

Fig. 2. Equilibrium with entry.

after the entry costs have been paid, $f = u/r$ firms have entered the market, and the shock y has been realized. From Section 2,

$$\Pi(r, y) = \begin{cases} \pi_1^*(ye^{-r}) & \text{if } r \leqslant \bar{q}_1(C_2/y), \\ \pi_1^*[ye^{-\bar{q}_1(C_2/y)}] & \text{if } \bar{q}_1(C_2/y) < r < \bar{q}_2(C_2/y), \\ \pi_2^*[ye^{-r}(1+r)] & \text{if } \bar{q}_2(C_2/y) \leqslant r. \end{cases}$$

Therefore, expected profits at the time of entry are

$$\tilde{\Pi}(r) = \int \Pi(r, y)\, dG(y), \tag{23}$$

and equilibrium is characterized by an r^* such that $\tilde{\Pi}(r^*) = C_e$. Lemma 2 describes properties of $\tilde{\Pi}$ that help establish the uniqueness of equilibrium in this environment.

Lemma 2. *Let $r_L = \bar{q}_1(C_2/y_L)$ and $r_H = \bar{q}_2(C_2/y_H)$. Expected profits $\tilde{\Pi}(r)$ are strictly increasing for $r < r_L$ and $r > r_H$, and constant for $r \in [r_L, r_H]$.*

Fig. 2 provides the basic intuition. Suppose that $G(y)$ is comprised of only two mass points, $y_L < y_H$, so that $\tilde{\Pi}(r)$ is a simple, convex combination of $\Pi(r, y_L)$ and $\Pi(r, y_H)$. The dotted line corresponds to $\Pi(r, y_L)$, the dashed line corresponds to $\Pi(r, y_H)$, and the solid line corresponds to $\tilde{\Pi}(r)$.

From the equilibrium characterization in Section 2, it is clear that $\Pi(r, y)$ is strictly increasing in r when either $r < \bar{q}_1(C_2/y)$ or $r > \bar{q}_2(C_2/y)$, and constant otherwise. Lemma 2, then, simply identifies the region $[r_L, r_H]$ in which *both* $\Pi(r, y_L)$ and $\Pi(r, y_H)$ are constant.[15] Note that if y_H is sufficiently large or y_L is sufficiently small, so that $\bar{q}_1(C_2/y_L) \geqslant \bar{q}_2(C_2/y_H)$, then $\tilde{\Pi}(r)$ is strictly increasing for all r. Regardless, *at worst* there exists a closed set with zero measure for which a unique equilibrium does not exist.

Proposition 2. *Let $\mathcal{C} = (0, 2 - C_2)$. Then for any $C_e \in \mathcal{C}$, there exists an equilibrium r^* such that $\tilde{\Pi}(r^*) = C_e$. Moreover, this equilibrium is unique almost everywhere; the set $\mathcal{C}' = \{C_e \in \mathcal{C}: \exists! r \text{ s.t. } \tilde{\Pi}^*(r) = C_e\}$ is an open set with full measure.*

3.3. Comparative statics

The uniqueness result in Proposition 2 allows for simple comparative statics. In the cross-section, the model suggests that more efficient matching should be associated with markets that discourage entry and encourage expansion. For example, from Fig. 2 one can see that a higher value of C_e decreases f^* and increases r^*.[16] Since $\partial\phi^*/\partial r \leqslant 0$ (from Proposition 1), this implies that a larger fraction of these firms will post multiple vacancies, ceteris paribus. However, the concentration of vacancies will also depend on the realized value of y, and the degree of convexity in the cost of posting vacancies (or, in an alternative setup, the concavity in the production technology). When the realized value of y is high relative to its expected value, or when the cost of posting multiple vacancies is low, one would expect firms to be expanding quickly (i.e. posting multiple vacancies) and matching to be more efficient.

In addition to these cross-sectional implications, the theory developed here suggests a dimension which may be important for accurately capturing the dynamics of the matching process: distinguishing between vacancy creation along the intensive and extensive margins. In other words, knowing only the number of new vacancies created during an economic expansion is insufficient to forecast the number of new matches formed; an economist would also need to know *how* these vacancies were created (by the entry of many small firms or by a few firms posting many vacancies). I discuss these implications below, by way of a simple numerical example.

For illustrative purposes, suppose that $u = 1$, $y_L = 0$, $y_H > 0$, and y is uniformly distributed over the interval $[0, y_H]$. Given C_e, C_2, y_H, and a realization y, an equilibrium is a pair (r^*, ϕ^*), with $f^* = u/r^*$ firms, $v^* = f^*[\phi^* + 2(1 - \phi^*)]$ vacancies, and $M^* = \hat{M}(u, v^*, \phi^*)$ matches. Fig. 3 plots market tightness, v^*/u, against the aggregate number of matches, M^*, for three different types of expansions. Recall that, in standard matching functions, market tightness is a sufficient statistic for the aggregate number of matches, so that there could be no distinction between these three expansions.

The curve AB plots M^* as the *realized* value of y increases; the values of u, C_e, C_2, and y_H are fixed so that $f^* = 1$ at all points along the curve, point A is the maximum realization of y such that $\phi^* = 1$, and point B is the minimum realization of y such that $\phi^* = 0$. In other words, this plots the response of M^* to changes in market tightness when all vacancies are created along the intensive margin. The curve AC, on the other hand, plots M^* as the *expected* value of y increases; the values of u, C_e, C_2 and y are set so that $\phi^* = 1$ at all points along the curve, and

[15] One interesting feature of Fig. 2 is that, for some values of r, $\Pi(r, y_L) > \Pi(r, y_H)$. The reason is that for a low realization of y, most firms post a single vacancy and there is little ex-post competition for workers. Therefore, wages remain low and profits are high. A crucial piece of this argument is that the efficiency of the matching function that arises in equilibrium is diminished when the distribution of vacancies is skewed towards single-vacancy firms.

[16] A lower level of expected productivity tends to have a similar effect.

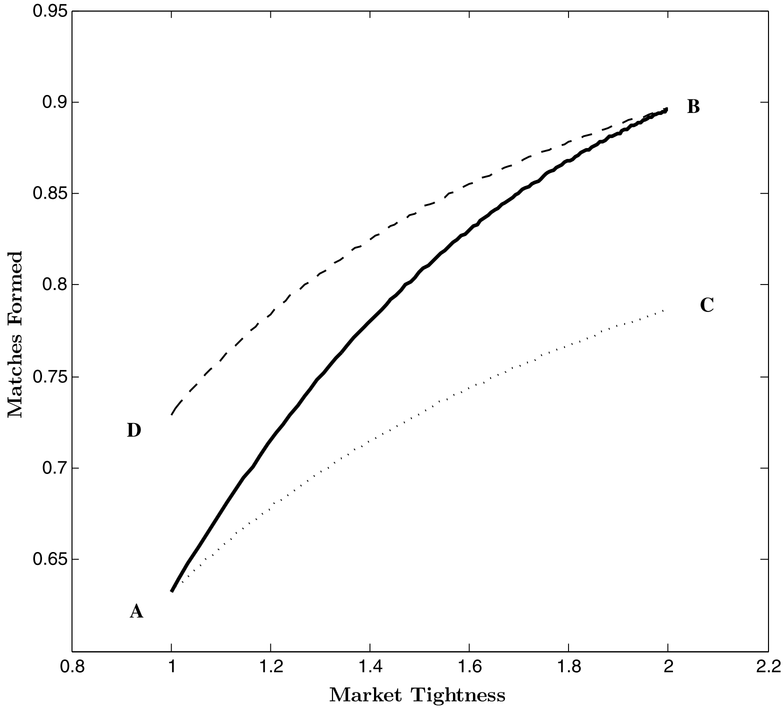

Fig. 3. Three expansions.

point C corresponds to the value of y_H such that $f^* = 2$. Therefore, AC represents the predicted change in the equilibrium number of matches when all vacancies are created along the extensive margin by single-vacancy firms. The curve DB also plots M^* as the expected value of y changes. Here the values of u, C_e, C_2 and y are set so that $\phi^* = 0$ at all points along the curve, and point D corresponds to the value of y_H such that $f^* = 1/2$, so that DB depicts the expected change in the number of matches when all vacancies are created along the extensive margin by two-vacancy firms.

Fig. 3 reveals that the source of an economic expansion has important implications for the efficiency of the matching process, and thus expected levels of unemployment and output. In particular, the matching function will be more responsive to changes in market tightness (i.e. exhibit a higher *elasticity*) during expansions caused by shocks that generate vacancy creation along the intensive margin.[17] In the example above, as realizations of y increase, a greater fraction of firms are posting two vacancies, coordination frictions ease, and the matching process becomes more efficient.

Both the cross-sectional and dynamic implications of the model are roughly consistent with a variety of stylized facts emerging from the data. In the cross-section, the model suggests that

[17] In fact, one can show that the matching technology exhibits constant returns to scale when vacancies are created along the extensive margin and increasing returns to scale when vacancies are created along the intensive margin. One could loosely interpret this finding to imply that the matching function exhibits constant returns in the long run, as in the data (see, e.g., Petrongolo and Pissarides [24]), but increasing returns to scale in the short run. I thank Chris Pissarides for a discussion that led to this insight.

the efficiency of the matching function should be greatest in markets with fewer firms, and in particular at those firms that are trying to expand quickly (by creating many vacancies). Using establishment-level data from the Japanese Ministry of Health, Labor, and Welfare, Ueno [30] constructs a measure of vacancy concentration across 47 different regions in Japan, and studies the relationship between the efficiency of the matching function and the concentration of vacancies. Consistent with the comparative statics above, she finds that the matching technology is more efficient and more sensitive to changes in market tightness in labor markets with a greater concentration of vacancies. Moreover, using establishment-level data from the Job Opening and Labor Turnover Survey (JOLTS), Davis, Faberman, and Haltiwanger [8] find that the vacancy yield (the probability that a vacancy is filled) rises steeply with firm growth rate.[18] Again, this indicates that the matching process is most efficient at firms posting multiple vacancies.

Over the cycle, Moscarini and Postel-Vinay [21] find that small firms create relatively more jobs in recessions, while large firms create relatively more jobs in expansions. To the extent that large firms are more likely to post multiple vacancies, the current model suggests that matching efficiency should be pro-cyclical. Cheremukhin and Echavarria [6] estimate shocks to the matching function over the cycle and find that these shocks are indeed pro-cyclical and important for explaining business cycle fluctuations. The model here provides micro-foundations for these as-yet-unidentified shocks to matching efficiency. In addition, the model could also potentially provide some insight into why directed search models fail to generate the observed elasticity of the job-finding rate (defined as M^*/u) with respect to market tightness. Using a reasonable calibration of the Burdett, Shi, and Wright [5] framework, Kaas [14] argues that the model-generated elasticity is approximately one third of that observed in the data. As the current model suggests, allowing firms to post multiple vacancies may help to resolve this discrepancy.[19]

All of these claims, of course, require much more serious empirical analysis, which is beyond the scope of the current paper; these facts are merely suggestive that the theory developed here could have important implications for a variety of issues in labor economics.

4. Heterogeneous firms

An impetus for early models of directed search, in particular that of Montgomery [19], was to explain wage differentials amongst homogeneous workers. The mechanism in such models was straightforward: by posting a higher wage, a firm could increase the expected number of applications it received, and thus increase the probability of filling its vacancy. Therefore, if firms were heterogeneous with respect to productivity, those firms with high productivity (i.e. a high cost of *not* filling a vacancy) would optimally choose to post high wages. With this mechanism, Montgomery [19] was able to generate wage differentials amongst homogeneous workers that were positively correlated with firms' profitability and hiring (or vacancy yield) rates. In the spirit of Montgomery's initial experiment, I now incorporate firm heterogeneity into the model with endogenous capacity, and illustrate that such a framework can easily and naturally generate

[18] Naturally, part of this relationship is tautological: those firms that grow must successfully fill their vacancies. However, the extreme non-linearity that they find suggests that there are efficiency gains at expanding firms that standard matching models may be unable to account for.

[19] Kaas [14] attempts to resolve the issue by introducing a mechanism by which workers can choose the intensity with which they apply to jobs, in the spirit of Albrecht et al. [3] and Galenianos and Kircher [9]. This approach can be viewed as complementary to the approach taken in this paper.

the positive relationship between wage differentials, profitability, vacancy yields, *and firm size* that is observed in the data.[20]

4.1. Extending the model

Consider again a fixed measure of firms, which can be normalized to one, that are heterogeneous with respect to their level of productivity: each firm $j \in [0,1]$ produces output y^j for each employed worker. This idiosyncratic productivity is distributed according to the cumulative density function $F(y^j)$, which is assumed to be continuously differentiable with full support over the interval $[0,\bar{y}] \subset \mathbb{R}_+$. I maintain the assumption that there exists a measure of homogeneous workers, and denote this measure by r to preserve the interpretation of r as the ratio of workers to firms. I also maintain the assumption that a firm can post a single vacancy at cost $C_1 = 0$ or two vacancies at cost $C_2 > 0$.[21]

The analysis is nearly identical to that in Section 2. In particular, given market utility U, the optimal queue length at a firm with productivity y^j and capacity k is the maximum of zero and the value of q_k^j that satisfies the firm's first-order condition. For $k = 1$ and $k = 2$, respectively, these first-order conditions yield:

$$U = y^j e^{-q_1^j}, \tag{24}$$

$$U = y^j e^{-q_2^j}\left(1 + q_2^j\right). \tag{25}$$

Let $Q_k^*(y^j, U)$ denote the unique value of q_k^j satisfying the relevant first-order condition whenever $y^j \geqslant U > 0$, and zero otherwise. Note that $Q_k^*(y^j, U) = 0$ if and only if $y^j \leqslant U$; if $y^j < U$, the firm cannot offer a worker expected payoff U without offering either $w_k^j > y^j$ (which is not profitable) or $q_k^j < 0$ (which is not feasible). For this reason, those firms with $y^j \leqslant U$ will be referred to as *inactive*.[22]

Again, equilibrium in the second stage requires $U_k(w_k^j, q_k^j) = U$, so that the wage corresponding to the optimal queue length for a firm with $k = 1$ and $k = 2$, respectively, is

$$w_1^j = \frac{y^j\{Q_1^*(y^j, U) e^{-Q_1^*(y^j,U)}\}}{1 - e^{-Q_1^*(y^j,U)}}, \tag{26}$$

$$w_2^j = \frac{y^j\{Q_2^*(y^j, U)[1 + Q_2^*(y^j, U)] e^{-Q_2^*(y^j,U)}\}}{2\{1 - e^{-Q_2^*(y^j,U)}\} - Q_2^*(y^j, U) e^{-Q_2^*(y^j,U)}}. \tag{27}$$

Given the optimal wage, I slightly abuse notation and define the maximal profits of a firm with capacity k and productivity y^j, given market utility U, as

$$\pi_1^*\left(y^j, U\right) = y^j\left\{1 - \left[1 + Q_1^*\left(y^j, U\right)\right] e^{-Q_1^*(y^j,U)}\right\}, \tag{28}$$

$$\pi_2^*\left(y^j, U\right) = y^j\left\{2 - e^{-Q_2^*(y^j,U)}\left\{2\left[1 + Q_2^*\left(y^j, U\right)\right] + Q_2^*\left(y^j, U\right)^2\right\}\right\} - C_2. \tag{29}$$

[20] There are many papers that have found that larger firms pay observationally equivalent workers higher wages; see, for example, Brown and Medoff [4]. For evidence that high wage firms are more profitable, see Abowd, Kramarz, and Margolis [1], among others. For evidence that wages are positively correlated with the queue lengths of applicants, and thus positively correlated with vacancy yield rates, see Holzer, Katz, and Krueger [12].

[21] Assuming $C_1 = 0$ is not entirely innocuous here. I point out below the implications of this assumption.

[22] I assume firms with $y^j = U$ remain inactive, though this is immaterial; there are a zero measure of such firms.

4.2. Equilibrium with heterogeneous firms

In order to characterize the equilibrium, I first establish that firms' optimal behavior can be summarized by a cut-off rule: for any market utility U, there will exist at most one level of productivity, which I denote $\tilde{y}$, such that firms with $y^j \in (\tilde{y}, \bar{y}]$ post two vacancies, firms with $y^j \in (U, \tilde{y}]$ post one vacancy, and firms with $y^j \in [0, U]$ are inactive.

Lemma 3. *For any $U \in (0, \bar{y})$, the gains from posting a second vacancy, $\pi_2^*(y^j, U) - \pi_1^*(y^j, U)$, are weakly increasing in y^j, and strictly increasing for any $y^j > U$. Moreover, there exists a $\hat{y} \in (0, \bar{y}]$ such that $\pi_2^*(y^j, U) - \pi_1^*(y^j, U) < 0$ for all $y < \hat{y}$.*

Lemma 3 establishes that there exists at most one value $\tilde{y} \in (0, \bar{y}]$ such that $\pi_2^*(\tilde{y}, U) = \pi_1^*(\tilde{y}, U)$, and that if such a $\tilde{y}$ does *not* exist then $\pi_2^*(y^j, U) < \pi_1^*(y^j, U)$ for all y^j. Thus, there are two potential types of equilibrium: either all active firms post a single vacancy, or there is a cutoff $\tilde{y} \in (0, \bar{y})$ such that firms post two vacancies if and only if $y > \tilde{y}$.[23] Lemma 4 shows that the gains from posting a second vacancy are decreasing in the level of market utility. This result will later help to partition the set of equilibria according to the values of the underlying parameters.

Lemma 4. *For $y^j \in [0, \bar{y}]$ and $U < y^j$, $\pi_2^*(y^j, U) - \pi_1^*(y^j, U)$ is strictly decreasing in U.*

Let $\bar{U}$ denote the value of U such that $\pi_1^*(\bar{y}, U) = \pi_2^*(\bar{y}, U)$. In words, $\bar{U}$ is the largest value of U such that some firms would at least weakly prefer to post two vacancies; Lemma 4 ensures that for any $U < \bar{U}$ there exists a strictly positive measure of firms that prefer to post two vacancies, and for any $U > \bar{U}$ all firms strictly prefer to post only a single vacancy. In addition, let $\hat{U}$ denote the value of U that satisfies

$$\int_0^{\bar{y}} Q_1^*\big(y^j, U\big)\, dF\big(y^j\big) = r.$$

In words, $\hat{U}$ is the market utility of workers when all firms choose to post a single vacancy. Note that the critical values $\bar{U}$ and $\hat{U}$ depend only on exogenous parameters such as r, C_2 and the distribution $F(y^j)$.

An equilibrium, then, can be characterized by a cut-off $\tilde{y}^* \in (0, \bar{y}]$ and a market utility U^* such that

$$\pi_1^*\big(\tilde{y}^*, U^*\big) \geqslant \pi_2^*\big(\tilde{y}^*, U^*\big), \quad \text{with equality if } \tilde{y}^* < \bar{y}, \tag{30}$$

and

$$\int_0^{\tilde{y}^*} Q_1^*\big(y^j, U^*\big)\, dF\big(y^j\big) + \int_{\tilde{y}^*}^{\bar{y}} Q_2^*\big(y^j, U^*\big)\, dF\big(y^j\big) = r. \tag{31}$$

[23] If $0 < C_1 < C_2$, there could exist a third type of equilibrium, in which all active firms post two vacancies. However, allowing for $C_1 > 0$ adds considerable complexity to the equilibrium characterization without substantially changing the basic insight.

The first condition requires that firms behave optimally, while the second condition ensures aggregate consistency.

Proposition 3. *If $\hat{U} \geqslant \bar{U}$ then there exists a unique equilibrium in which all firms post a single vacancy, so that $\tilde{y}^* = \bar{y}$ and $U^* = \hat{U}$. If $\hat{U} < \bar{U}$ then there exists a unique equilibrium in which some firms post a single vacancy and others post two vacancies, so that $\tilde{y}^* < \bar{y}$ and $U^* < \bar{U}$.*

To provide some intuition for Proposition 3, note that $\bar{U}$ is decreasing in C_2 and increasing in $\bar{y}$, while $\hat{U}$ is decreasing in r. Therefore, an equilibrium with only single-vacancy firms is likely when C_2 is large, $\bar{y}$ is small, and/or r is small. This is natural: if it costly to post a second vacancy, if there are few very productive firms, or if there are few workers, then the benefits of posting a second position will not justify the costs.

4.3. Productivity, wages, hiring, firm size, and profits

Given the results above, it is straightforward to establish that firms with multiple vacancies are more profitable than firms with a single vacancy. Moreover, one can show that $Q_1^*(\tilde{y}^*, U^*) < Q_2^*(\tilde{y}, U^*)/2$ and $\partial Q_k^*/\partial y^j > 0$ for $k \in \{1, 2\}$ and $y^j > U^*$, so that multi-vacancy firms also receive more applications per position and fill their vacancies with greater probability than single vacancy firms. Finally, though wages are *not* monotonically increasing in productivity (and thus in firm size), I illustrate below that the model can easily generate the result that the *average* wage at multi-vacancy firms is greater than that at single-vacancy firms.

Consider the following simple numerical example. Suppose that y^j is distributed uniformly across the interval [0, 1], so that $F(y^j) = y^j$ and $\bar{y} = 1$, and also that $r = 1$ and $C_2 = .1$. The solid lines in Figs. 4 and 5 depict the equilibrium values of wages and the vacancy yield, respectively, for different values of y^j. In this equilibrium, $\tilde{y}^* = .5$ and $U^* = .316$.

First, note that more productive firms do not necessarily offer higher wages when they can also offer a greater hiring probability through posting multiple vacancies. In particular, those multi-vacancy firms with productivity slightly above $\tilde{y}$ offer lower wages than those single-vacancy firms with productivity slightly below $\tilde{y}$. However, so long as there is sufficient dispersion in productivity (as there is in the example here), the average wage offered by firms with two vacancies (the dashed line) is greater than the average wage offered by firms with one vacancy (the dotted line).

Second, notice that the vacancy yield jumps when firms switch from posting a single vacancy to two vacancies. Now, directed search models in general are going to predict a positive relationship between productivity, queue length, and vacancy yield; this is a direct result of allowing the wage to play an allocative role in the workers' application decision. However, in comparison to a model that assumed an exogenous number of vacancies per firm, the model here predicts a greater disparity between the vacancy yield for firms with one vacancy and with multiple vacancies.[24]

5. Conclusion

The arrival of establishment-level data promises to offer a bevy of new stylized facts with regard to how establishment-specific characteristics affect observable outcomes, such as the

[24] Again, this observation could further help in understanding the highly non-linear relationship between the vacancy yield and employer growth documented in Davis, Faberman, and Haltiwanger [8].

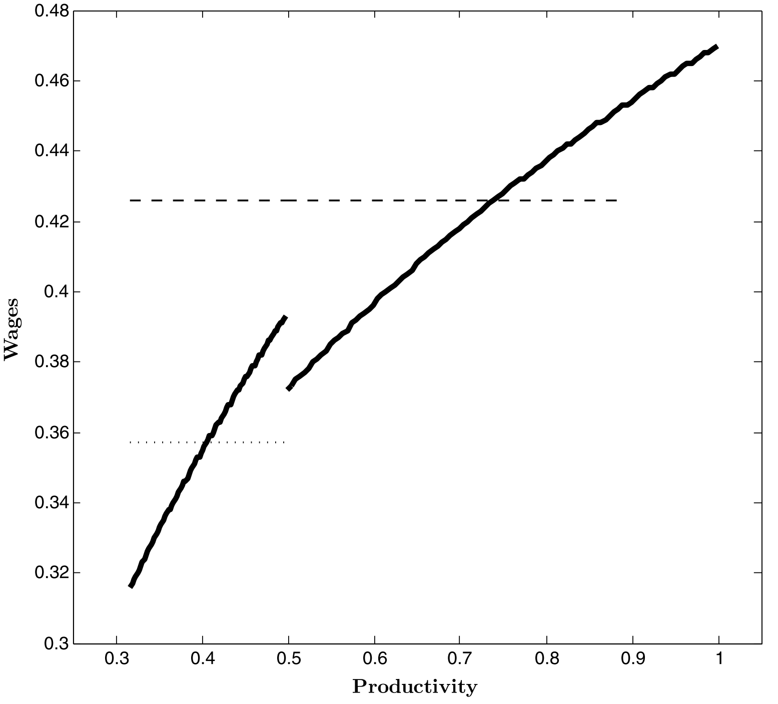

Fig. 4. Wages.

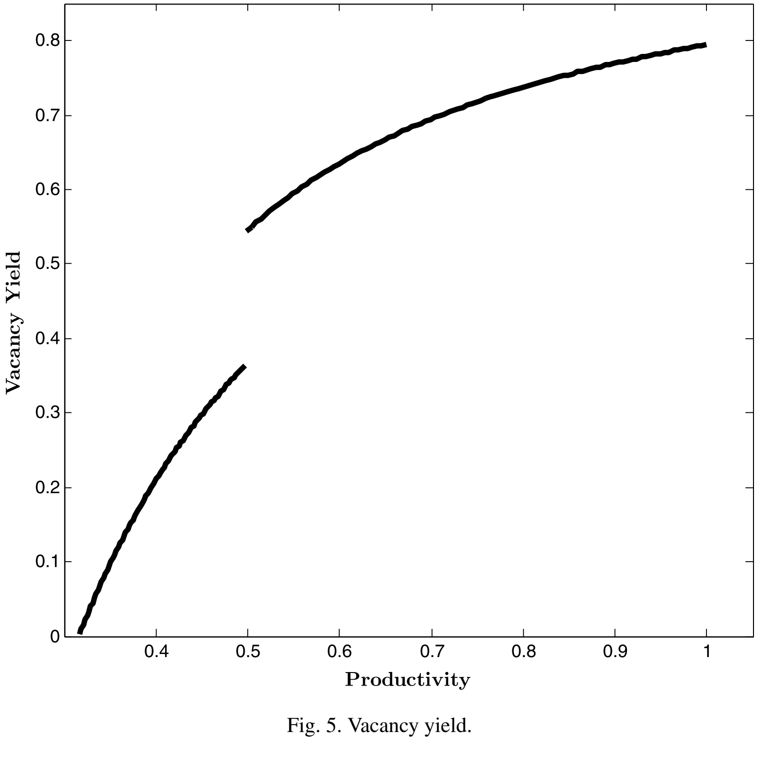

Fig. 5. Vacancy yield.

efficiency of the matching process (job-finding and job-filling rates) and wages. As a result, developing theoretical models that dig deeper into the relationship between these outcomes and the behavior of individual workers and firms has become an active area of research. One particularly appealing framework is that of directed search, since it explicitly models both wage formation and matching, and can accommodate the introduction of various types of heterogeneity on both sides of the market. Naturally, early formulations of the directed search environment made a variety of strict assumptions on workers and firms, thus limiting the scope of the model as an analytical tool. In this paper, I introduce a modeling device that provides a distinction between a firm and a vacancy. I show that this distinction is important, as the matching technology depends crucially on the distribution of vacancies across firms. This finding implies a rich set of testable predictions, both across markets and over time. I also show that with a simple extension, the model can account for the observed relationship between firm size, profitability, hiring, and wages in a very natural way.

There are several other extensions of this framework that could yield interesting results. For example, allowing workers to apply to multiple firms, one could study the interaction between the extent to which firms relax coordination frictions (by opening multiple vacancies) and the extent to which workers relax these frictions (by applying to multiple firms). Alternatively, embedding the current framework into a dynamic model (in the spirit of Menzio and Shi [17]) could provide further insights into the evolution of firm size over time. These extensions are left for future work.

Appendix A

A.1. Proofs

Proof of Lemma 1. Plugging (16) into (17) and simplifying yields (18). The right-hand side of (18) is constant and lies strictly within the interior of the unit interval, while the left-hand side is decreasing in q_2, equal to 1 at $q_2 = 0$, and converges to 0 as $q_2 \to \infty$. Therefore, there is a unique root. □

The following lemma establishes a property of interior equilibria that allows me to characterize the parameter space in which they exist.

Lemma 5. *In any interior equilibrium,* $q_1 < r < q_2$.

Proof. First, it should be obvious that it cannot be the case that both q_1 and q_2 are either strictly greater than r or strictly less than r. Therefore, it remains to be shown that $q_2 > q_1$ in any interior equilibrium. However, note that if $q_1 \geqslant q_2$, then $e^{-q_1} \leqslant e^{-q_2} < (1+q_2)e^{-q_2}$, so that (16) could not hold. □

Proof of Proposition 1. I begin by establishing that $r \leqslant \bar{q}_1(c)$ is a necessary and sufficient condition for the existence of an equilibrium with $\phi^* = 1$. First, suppose $r \leqslant \bar{q}_1(c)$ and define $\tilde{q}_2$ as the value of q_2 that satisfies $r = q_2 - \log(1+q_2)$. Then

$$
\begin{aligned}
r \leqslant \bar{q}_1(c) \quad &\Leftrightarrow \quad \tilde{q}_2 - \log(1+\tilde{q}_2) \leqslant \bar{q}_2(c) - \log\bigl[1+\bar{q}_2(c)\bigr] &(32)\\
&\Leftrightarrow \quad \tilde{q}_2 \leqslant \bar{q}_2(c) &(33)\\
&\Leftrightarrow \quad e^{-\tilde{q}_2}\bigl[1+(1+\tilde{q}_2)\log(1+\tilde{q}_2)\bigr] \geqslant 1 - C_2/y. &(34)
\end{aligned}
$$

Consider the problem of a single firm when $q_1 = r$ and $U = ye^{-r}$. Profits from opening one or two vacancies are given by

$$\pi_1^*(U) = y\left[1 - e^{-r}(1+r)\right],$$
$$\pi_2^*(U) = y\left\{2 - e^{-\tilde{q}_2}\left[2(1+\tilde{q}_2) + \tilde{q}_2^2\right]\right\} - C_2,$$

respectively, and $\pi_1^*(U) \geqslant \pi_2^*(U)$ if and only if (34) is true. Therefore $r \leqslant \bar{q}_1(c)$ implies that there exists an equilibrium with $\phi^* = 1$. Now suppose there exists an equilibrium with $\phi^* = 1$. Then clearly (34) must hold, so that

$$\tilde{q}_2 \leqslant \bar{q}_2(c) \quad \Leftrightarrow \quad \tilde{q}_2 - \log(1+\tilde{q}_2) \leqslant \bar{q}_2(c) - \log\left[1 + \bar{q}_2(c)\right]$$
$$\Leftrightarrow \quad r \leqslant \bar{q}_1(c).$$

A symmetric argument can be used to establish that $r \geqslant \bar{q}_2(c)$ is a necessary and sufficient condition for the existence of an equilibrium with $\phi^* = 0$. Given Lemmas 1 and 5, this establishes the existence and uniqueness of each of the three different type of equilibria. The expressions in (20) and (21) follow directly from the observation that in all interior equilibria $\bar{q}_1(c) = (r\sigma^*)/\phi^*$ and $\bar{q}_2(c) = [r(1-\sigma^*)]/(1-\phi^*)$.

To conduct comparative statics, note that $\partial\bar{q}_1/\partial c = [\bar{q}_2/(1+\bar{q}_2)](\partial\bar{q}_2/\partial c)$ and

$$\frac{\partial\bar{q}_2}{\partial c} = \frac{1}{e^{-\bar{q}_2}\bar{q}_2\log(1+\bar{q}_2)} > 0.$$

Given this result and Lemma 5, it is easy to see

$$\frac{\partial\phi^*}{\partial r} = \frac{-1}{\bar{q}_2 - \bar{q}_1} < 0,$$
$$\frac{\partial\sigma^*}{\partial r} = \frac{-\bar{q}_1\bar{q}_2}{r^2(\bar{q}_2 - \bar{q}_1)} < 0,$$
$$\frac{\partial\phi^*}{\partial c} = \frac{\frac{\partial\bar{q}_1}{\partial c}(\bar{q}_2 - r) + \frac{\partial\bar{q}_2}{\partial c}(r - \bar{q}_1)}{(\bar{q}_2 - \bar{q}_1)^2} > 0,$$
$$\frac{\partial\sigma^*}{\partial c} = \frac{\frac{\partial\bar{q}_1}{\partial c}(\bar{q}_2 - r)\bar{q}_2 + \frac{\partial\bar{q}_2}{\partial c}(r - \bar{q}_1)\bar{q}_1}{r^2(\bar{q}_2 - \bar{q}_1)^2} > 0,$$

where the arguments of $\bar{q}_1$ and $\bar{q}_2$ have been suppressed for notational convenience. □

Proof of Result 1. For any $u > 0$, $v > 0$, and $\phi \in (0, 1)$,

$$\hat{M}(u, v, \phi) = f\left[\phi m_1(q_1) + (1-\phi)m_2(q_2)\right], \tag{35}$$

where $f = v/(2-\phi)$, $q_1 = [u(2-\phi)\Sigma(\phi)]/(v\phi)$, and $q_2 = \{u(2-\phi)[1-\Sigma(\phi)]\}/[v(1-\phi)]$. Differentiating (35) with respect to ϕ yields

$$\frac{\partial\hat{M}}{\partial\phi} = f\left[\phi m_1'(q_1)\frac{\partial q_1}{\partial\phi} + m_1(q_1) + (1-\phi)m_2'(q_2)\frac{\partial q_2}{\partial\phi} - m_2(q_2)\right]$$
$$+ \frac{\partial f}{\partial\phi}\left[\phi m_1(q_1) + (1-\phi)m_2(q_2)\right].$$

Note that $\partial f/\partial\phi = v/[(2-\phi)^2] = f/(2-\phi)$, and that $m_1'(q_1) = e^{-q_1}$ is equal to $m_2'(q_2) = e^{-q_2}(1+q_2)$ because of (19). This implies

$$\frac{\partial \hat{M}}{\partial \phi} = f\left\{\frac{[-q_2 e^{-q_2}] + e^{-q_2}(1+q_2)[\phi\frac{\partial q_1}{\partial \phi} + (1-\phi)\frac{\partial q_2}{\partial \phi}]}{2-\phi}\right\}. \tag{36}$$

Since

$$\frac{\partial q_1}{\partial \phi} = \frac{u}{v}\left[\frac{\phi(2-\phi)\Sigma'(\phi) - 2\Sigma(\phi)}{\phi^2}\right],$$
$$\frac{\partial q_2}{\partial \phi} = \frac{u}{v}\left[\frac{[1-\Sigma(\phi)] - (1-\phi)(2-\phi)\Sigma'(\phi)}{(1-\phi)^2}\right],$$

(36) can be further reduced to

$$\frac{\partial \hat{M}}{\partial \phi} = \frac{f}{2-\phi}\left[-q_2 e^{-q_2} + e^{-q_2}(1+q_2)\left(\frac{q_2 - 2q_1}{2-\phi}\right)\right].$$

If $q_2 - 2q_1 \leqslant 0$, clearly $\partial \hat{M}/\partial \phi < 0$. If $q_2 - 2q_1 > 0$, then

$$\frac{\partial \hat{M}}{\partial \phi} \leqslant \frac{f}{2-\phi}\left[-q_2 e^{-q_2} + e^{-q_2}(1+q_2)(q_2 - 2q_1)\right]$$
$$= \frac{-f e^{-q_2}}{2-\phi}\left[q_2^2 + 2q_2 - q(1+q_2)\log(1+q_2)\right].$$

One can easily establish that the term in brackets is a strictly increasing function of q_2 and converges to zero as $q_2 \to 0^+$, so that it is strictly positive for all $q_2 > 0$, and thus $\partial \hat{M}/\partial \phi < 0$. □

Proof of Lemma 2. It will be convenient to define the functions $\bar{c} = \bar{q}_1^{-1}$ and $\underline{c} = \bar{q}_2^{-1}$. Given r, for all $c \geqslant \bar{c}(r)$ ($c \leqslant \bar{c}(r)$) equilibrium is characterized by all firms posting one vacancy (two vacancies). I will first establish that both $\underline{c}(r)$ and $\bar{c}(r)$ are strictly increasing in r. Recall that $\tilde{q}_2$ was defined as the value of q_2 such that $e^{-r} = e^{-q_2}(1+q_2)$.

$$\frac{\partial \underline{c}}{\partial r} = re^{-r}\log(1+r) > 0,$$
$$\frac{\partial \bar{c}}{\partial r} = -e^{-\tilde{q}_2}\left\{\frac{\partial \tilde{q}_2}{\partial r}\left[\tilde{q}_2(1+r) - \tilde{q}_2^2\right] - (1+\tilde{q}_2)\right\}$$
$$= -e^{-\tilde{q}_2}\left\{\frac{e^{-r}}{\tilde{q}_2 e^{-\tilde{q}_2}}\left[\tilde{q}_2(1+r) - \tilde{q}_2^2\right] - (1+\tilde{q}_2)\right\}$$
$$= -e^{-r}(1+r-\tilde{q}_2) + e^{-\tilde{q}_2}(1+\tilde{q}_2)$$
$$= e^{-r}(\tilde{q}_2 - r) > 0,$$

because $\tilde{q}_2 > r\ \forall r > 0$.

It is clear that profits are weakly increasing in r for any realization of y. I will establish that for all values of r such that $r < r_L$ or $r > r_H$, profits are strictly increasing for *some* realization $y \in \text{supp}(G)$, and for all values of $r_L < r < r_H$ profits are constant for all $y \in \text{supp}(G)$. Now consider any r', r'' such that $r'' > r'$. Below I derive the equilibrium profits for r' and r'' corresponding to different realizations of y. In order to compare $\Pi(r', y)$ and $\Pi(r'', y)$ for any value of $y \geqslant 0$, given C_2, there are (at most) five relevant regions:

1. If $y \in [C_2/\underline{c}(r'), \infty]$, $\Pi(r', y) = \pi_2^*[ye^{-r'}(1+r')] < \pi_2^*[ye^{-r''}(1+r'')] = \Pi(r'', y)$.
2. If $y \in [C_2/\underline{c}(r''), C_2/\underline{c}(r')]$, $\Pi(r', y) = \pi_2^*\{ye^{-\bar{q}_2(C_2/y)}[1+\bar{q}_2(C_2/y)]\} < \pi_2^*[ye^{-r''}(1+r'')] = \Pi(r'', y)$ since $C_2/y < \underline{c}(r'') \Leftrightarrow r'' > \bar{q}_2(C_2/y)$.
3. If $y \in [C_2/\bar{c}(r'), C_2/\underline{c}(r'')]$, $\Pi(r', y) = \pi_2^*\{ye^{-\bar{q}_2(C_2/y)}[1+\bar{q}_2(C_2/y)]\} = \Pi(r'', y)$.
4. If $y \in [C_2/\bar{c}(r''), C_2/\bar{c}(r')]$, $\Pi(r', y) = \pi_1^*[ye^{-r'}] < \pi_1^*[ye^{-\bar{q}_1(C_2/y)}] = \Pi(r'', y)$ since $C_2/y > \bar{c}(r') \Leftrightarrow r' < \underline{r}(C_2/y)$.
5. If $y \in (0, C_2/\bar{c}(r'')]$, $\Pi(r', y) = \pi_1^*[ye^{-r'}] < \pi_1^*[ye^{-r''}] = \Pi(r'', y)$.

It is clear, then, that $\Pi(r'', y) \geqslant \Pi(r', y)\ \forall y$, with strict inequality for values of y such that $C_2/y < \underline{c}(r'')$ and $C_2/y > \bar{c}(r')$.

Consider some value r' such that $r' < r_L$. Then $\Pi(r', y_L) = \pi_1^*(ye^{-r'})$ and subsequently $\partial\tilde{\Pi}(r)/\partial r > 0$. Similarly, if r'' is such that $r'' > r_H$, then $\Pi(r'', y_H) = \pi_2^*[ye^{-r''}(1+r'')]$ and subsequently $\partial\tilde{\Pi}(r)/\partial r > 0$. However, for any $r \in [r_L, r_H]$, $\Pi(r, y) = \pi_1^*[ye^{-\bar{q}_1(C_2/y)}] = \pi_2^*\{ye^{-\bar{q}_2(C_2/y)}[1+\bar{q}_2(C_2/y)]\}\ \forall y_L \leqslant y \leqslant y_H$. Therefore, in this region, expected profits are independent of r; that is,

$$\tilde{\Pi}(r) = \int \pi_1^*\big[ye^{-\bar{q}_1(C_2/y)}\big]\,dG(y). \qquad \square$$

Proof of Proposition 2. Existence follows immediately from the fact that $\tilde{\Pi}$ is continuous, $\lim_{r\to 0}\tilde{\Pi}(r) = 0$, and $\lim_{r\to\infty}\tilde{\Pi}(r) = 2 - C_2$. The uniqueness result follows immediately from the fact that $\tilde{\Pi}$ is constant over a single interval, and strictly increasing elsewhere. $\square$

Proof of Lemma 3. To economize on notation, let $q_k^j \equiv Q_k^*(y^j, U)$ for $k \in \{1,2\}$. First note that

$$\begin{aligned}\frac{\partial}{\partial y^j}\big[\pi_2^*(y^j,U) - \pi_1^*(y^j,U)\big] &= \big[2 - e^{-q_2^j}\big(2+q_2^j\big)\big] - \big[1 - e^{-q_1^j}\big]\\ &= 1 - e^{-q_2^j} - \big[e^{-q_2^j}\big(1+q_2^j\big) - e^{-q_1^j}\big]\\ &= 1 - e^{-q_2^j} \geqslant 0\end{aligned}$$

for all $q_2^j \geqslant 0$, with strict inequality if $q_2^j > 0$, where I have used the fact that $e^{-q_2^j}(1+q_2^j) = e^{-q_1^j} = U/y^j$. This establishes the first fact, and the second fact follows immediately since $\pi_2^*(0,U) = -C_2 < 0 = \pi_1^*(0,U)$. $\square$

Proof of Lemma 4. This follows immediately from the fact that, for $y^j > U$,

$$\frac{\partial}{\partial U}\big[\pi_2^*(y^j,U) - \pi_1^*(y^j,U)\big] = q_1^j - q_2^j < 0. \qquad \square \tag{37}$$

Proof of Proposition 3. If $\hat{U} \geqslant \bar{U}$, then $\pi_2^*(\bar{y},U) - \pi_1^*(\bar{y},U) \leqslant 0$ from Lemma 4. Therefore $\pi_2^*(y^j,U) \leqslant \pi_1^*(y^j,U)\ \forall y^j \in [0,\bar{y}]$ by Lemma 3, so that $\tilde{y}^* = \bar{y}$ satisfies (30), and by construction $U^* = \hat{U}$ satisfies (31). To see that this is the only equilibrium, suppose towards a contradiction that there exists another equilibrium with $\tilde{y}^* < \bar{y}$. When $\tilde{y}^* = \bar{y}$, the market utility is $U^* = \hat{U}$. One can implicitly differentiate (31) using Leibniz's rule to get that

$$\frac{\partial U^*}{\partial \tilde{y}^*} = \frac{-[Q_2^*(\tilde{y}^*, U^*) - Q_1^*(\tilde{y}^*, U^*)]}{\int_0^{\tilde{y}^*} \frac{1}{y^j e^{-Q_1^*(y^j, U^*)}} dF(y^j) + \int_{\tilde{y}^*}^{\bar{y}} \frac{1}{y^j e^{-Q_2^*(y^j, U^*)} Q_2^*(y^j, U^*)} dF(y^j)} < 0. \tag{38}$$

Therefore, if $\exists \tilde{y}^* < \bar{y}$, the associated market utility $U^* \geqslant \hat{U} \geqslant \bar{U}$. But $U^* \geqslant \bar{U} \Rightarrow \pi_2^*(y^j, U^*) < \pi_1^*(y^j, U^*)\ \forall y^j \in (0, \bar{y}) \Rightarrow \nexists \tilde{y}^* \in (0, \bar{y})$ such that $\pi_2^*(\tilde{y}^*, U^*) = \pi_1^*(\tilde{y}^*, U^*)$, a contradiction.

Now suppose that $\hat{U} < \bar{U}$. I will show that there exists a pair $(\tilde{y}^*, U^*) \in (0, \bar{y}) \times (0, \bar{U})$ that satisfies the equilibrium conditions (30) and (31), and that this pair is unique. To do so, it will be convenient to define $\Upsilon_F : U \mapsto \tilde{y}$ as the implicit function in Eq. (30), which represents the marginal firm's indifference condition. Likewise, define $\Upsilon_A : U \mapsto \tilde{y}$ as the implicit function in Eq. (31), which represents the aggregate consistency requirement. I will show that $\hat{U} < \bar{U} \Rightarrow \exists! U^*$ such that $\Upsilon_F(U^*) = \Upsilon_A(U^*) \equiv \tilde{y}^* < \bar{y}$.

Note that (i) $\lim_{U \to 0} \Upsilon_F(U) = C_2$, (ii) $\lim_{U \to \bar{U}} \Upsilon_F(U) = \bar{y}$, and (iii) Υ_F is increasing in U:

$$\frac{\partial \Upsilon_F(U)}{\partial U} = \frac{Q_2^*(\tilde{y}, U) - Q_1^*(\tilde{y}, U)}{1 + e^{-Q_1^*(\tilde{y}, U)} - e^{-Q_2^*(\tilde{y}, U)}[2 + Q_2^*(\tilde{y}, U)]} > 0. \tag{39}$$

Also note that (i) $\lim_{U \to \hat{U}} \Upsilon_A(U) = \bar{y}$ by definition, and (ii) Υ_A is decreasing in U, as established in (38) above. Since F is assumed to be continuous, I know that both Υ_F and Υ_A are continuous as well. Therefore, since $\hat{U} < \bar{U}$, it is clear that Υ_F and Υ_A intersect at a unique value $U^* \in (\hat{U}, \bar{U})$ and that $\Upsilon_F(U^*) = \Upsilon_A(U^*) < \bar{y}$. □

A.2. Discussion of assumptions

In this subsection, I will show that the results concerning matching and efficiency do not depend on the assumption that a firm with two positions pays both workers the same wage. I will also illustrate how very similar results could be derived under the alternative assumption that there are decreasing returns to scale in the production technology and constant returns in the cost of posting vacancies. To see both of these points, suppose $y_1 > y_2 > 0$, $C_1 = C_2 \equiv C > 0$, and consider the profit maximization problem of a firm with two vacancies that offers wage w' to the first worker hired, and w'' to the second. With a little algebra, the problem can be written

$$\max_{q_2, w', w''} \left(1 - e^{-q_2}\right)\left(y_1 - w'\right) + \left[1 - e^{-q_2}(1 + q_2)\right]\left(y_2 - w''\right) - 2C \tag{40}$$

$$\text{subject to } \left[\left(1 - e^{-q_2}\right)/q_2\right] w' + \left\{\left[1 - e^{-q_2}(1 + q_2)\right]/q_2\right\} w'' = U. \tag{41}$$

Again, this can be reformulated as a strictly concave problem in q_2, with the unique solution satisfying the condition

$$e^{-q_2}(y_1 + q_2 y_2) = U. \tag{42}$$

Note that the queue length satisfying (42) can be achieved by a continuum of pairs (w', w'') satisfying the constraint (41), and in particular the pair

$$w' = w'' \equiv \tilde{w}_2 = \frac{q_2 e^{-q_2}(y_1 + q_2 y_2)}{2(1 - e^{-q_2}) - q e^{-q_2}}. \tag{43}$$

Moreover, note that all of the results concerning matching and efficiency will depend on the equilibrium queue length, and not on the pair (w', w''). Finally, consider the equilibrium in which the wage at two-vacancy firms is described by $\tilde{w}_2$. This wage is the analog of (13); following the same steps that were employed in Section 2, one can easily derive all of the analogous results for the case of $y_1 > y_2 > 0$ and $C_1 = C_2 \equiv C > 0$.

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